Attachment D — Compactness Closure for the Full Frozen Axis-Fold Window

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Attachment D to Structural Classification of the Exterior-Exponent Collapse Mechanism (Paper I), 2026. Continuation of Attachment C. All statements concern the operator and domain fixed in §1 of Attachment C. Constants may depend on its fixed positive parameters and on the nonzero real parameter γ. Bracketed citations refer to the reference list of Paper I.

1. Statement and dependencies [EXACT]

Write $t = |\tau|$, $b = d\sqrt{\kappa_E}$ and $h = \rho_t = (\nu b^3/(7\gamma^4 t))^{1/7}$. The remaining input from Propositions 9.55–9.57 is the following: if $t_n \to \infty$ and $\| z_n \|_{\mathcal{H}} + \| f_n \|_{\mathcal{H}}$ is bounded, where $f_n = \mathcal{R}(\tau_n) z_n$, then the corrected amplitudes tend to zero outside each fixed inner fraction of h_n, and

$$\| \mathbf{1}_{r \geq \varepsilon h_n} z_n \|_{\mathcal{H}} \to 0 \quad (\varepsilon > 0).$$

We close the remaining inner region below. The new ingredients are an explicit cubic inverse at the axis and an interval estimate for the exact six-component radial system. No coefficient, primitive endpoint or original boundary condition is removed from the solution being estimated.

2. An exact differentiated operator [EXACT]

Put $E = (\sqrt{\kappa_E}/2) re$, $y = (E, u)^T$, and $\widehat{f} = ((\sqrt{\kappa_E}/2) r f_e, f_u)^T$. This is an isometry from $\mathcal{H}$ to $L^2(rdr)^2$. Define

$$D = \begin{pmatrix} 0 & \partial_r \\ -\partial_r - r^{-1} & 0 \end{pmatrix}, \qquad D_{\mathrm{m}} = D - 2r^{-1} E_{21}.$$

Here E_{ij} denotes a matrix unit, not the field E. Direct differentiation of both primitives gives

$$D^2 = \mathrm{diag}(-\Delta_2 + r^{-2}, -\Delta_2), \qquad \widehat{D\mathcal{A}_0} = bI + \gamma r^2 D_{\mathrm{m}}.$$

Consequently every original resolvent state satisfies, in distributions,

$$\boxed{(\nu D^3 + i\tau b) y + i\tau\gamma r^2 D_{\mathrm{m}} y = D\widehat{f}.}$$

The shear is needed for this identity: its differentiated upper row cancels the derivative of the upper tilt coefficient. Differentiating provides a necessary identity for actual solutions; it is not used to prescribe solutions of a new boundary problem.

On the whole radial half-line, the regular realization of D is self-adjoint. One direct verification embeds (E, u) as a radial vector field and a scalar on $\mathbb{R}^2$: the operator is $(V, u) \mapsto (\nabla u, -\mathrm{div} V)$, whose Fourier symbol is Hermitian. The radial subspace is invariant, and D^2 has the displayed radial expressions. Fourier functional calculus therefore gives

$$B_\tau = (\nu D^3 + i\tau b)^{-1}, \qquad \| B_\tau D^j \| \leq C_j t^{j/3 - 1}, \quad 0 \leq j \leq 3.$$

Indeed the scalar multiplier has modulus $|\lambda|^j/(\nu^2\lambda^6 + t^2 b^2)^{1/2}$ for real λ. These bounds also specify its action on derivative forcing, so no derivative of f is assumed to lie in L^2.

3. Exclusion of the axis core [EXACT]

Proposition 9.58. For every fixed $B > 0$, all sufficiently large t and every original-domain state,

$$\| \mathbf{1}_{r \le Bt^{-1/6}} z \|_{\mathcal{H}} \le C_B \left(t^{-2/3} \| f \|_{\mathcal{H}} + t^{-1/12} \| z \|_{\mathcal{H}} \right).$$

Proof, first the axis. Let $a = \alpha t^{-1/6}$ and choose a smooth radial χ equal to one below a, zero above $2a$, with $|\chi^{(j)}| \le C_j a^{-j}$. Extend χy by zero to the whole half-line. Put $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The exact identities

$$\chi D\widehat{f} = D(\widehat{\chi f}) - J\chi'\widehat{f}, \qquad \chi r^2 D_{\mathrm{m}} y = D(r^2 \chi y) - 2r\chi E_{12} y - r^2 \chi' J y$$

allow use of B_τ without differentiated forcing. Also

$$[D^3, \chi] y = D^2(C_2 y) + D(C_1 y) + C_0 y,$$

where the coefficients are supported in $[a, 2a]$ and $\| C_j \|_\infty \le C a^{j-3}$. For completeness,

$$C_2 = 3J\chi', \qquad C_1 = \mathrm{diag}(3\chi'' - \chi'/r, 3\chi'' + \chi'/r),$$

$$C_0 = \begin{pmatrix} 0 & -\chi''' + \chi''/r - \chi'/r^2 \\ \chi''' + \chi''/r - \chi'/r^2 & 0 \end{pmatrix}.$$

Apply B_τ to the localized differentiated equation. Its norm bounds give

$$\| \chi y \| \le C(t^{-2/3} + t^{-1}a^{-1}) \| \widehat{f} \| + C \sum_{j=0}^{2} t^{j/3-1} a^{j-3} \| y \| + Ct^{1/3} a^2 \| \chi y \| + Ca \| y \| .$$

Choose a fixed $\alpha > 0$ small enough to absorb $C\alpha^2 \| \chi y \|$. This proves

$$\| \mathbf{1}_{r \le \alpha t^{-1/6}} y \| \le C_\alpha (t^{-2/3} \| \widehat{f} \| + t^{-1/6} \| y \|).$$

The rest of a fixed core. On $\alpha t^{-1/6}/2 \le r \le (B+1)t^{-1/6}$ use the flat coordinates $U = r^{1/2} y$, $F = r^{1/2} \widehat{f}$. The principal constant-coefficient operator at radius ρ is

$$P_{\tau,\rho} = \nu (J\partial_r)^3 + i\tau(bI + \gamma\rho^2 J\partial_r).$$

For $\rho t^{1/6}$ in a fixed compact interval its whole-line Fourier inverse satisfies

$$\| P_{\tau,\rho}^{-1} \partial_r^j \| \le C_B t^{j/3-1}, \qquad 0 \le j \le 3.$$

To see uniformity, write a Fourier eigenvalue of $J\partial_r$ as $t^{1/3}s$. Dividing the multiplier by t gives $\nu s^3 + i\mathrm{sgn}(\tau)(b + qs)$, with $q = \gamma\rho^2 t^{1/3}$ in a fixed compact set. Its squared modulus is $\nu^2 s^6 + (b+qs)^2$. It has no zero: vanishing of its first term forces $s = 0$, where the second is b^2. Compactness for bounded s and cubic growth for large s prove the bound.

Choose cutoffs on intervals of length $\ell = t^{-1/4}$, with bounded overlap and derivative bounds $C_j \ell^{-j}$, whose squares cover $[\alpha t^{-1/6}, Bt^{-1/6}]$. Each localized flat state obeys the constant-coefficient equation plus the following errors, after writing derivatives to the left of their coefficient times the state:

Error	Bound after the constant inverse
Radial lower terms $r^{-1}\partial_r^2, r^{-2}\partial_r, r^{-3}$	$C_B t^{-1/6}$ times the localized state norm
Tilt variation $i\tau\gamma(r^2 - \rho^2)J\partial_r$	$C_B t^{1/3} t^{-1/6} \ell = C_B t^{-1/12}$ times that norm
Cubic cutoff commutators	$C_B t^{-1/3} \ell^{-1} = C_B t^{-1/12}$ times the state norm on the cutoff support, with smaller second and third derivative terms
Mass cutoff commutator $i\tau\gamma r^2 J\chi'$	$C_B t^{-1/3} \ell^{-1} = C_B t^{-1/12}$ times the state norm on its support

Error	Bound after the constant inverse
Remaining mass term $O(\tau r)$	$C_B t^{-1/6}$ times the localized state norm

The derivative forcing contributes at most $C_B t^{-2/3}$ times the forcing norm on the cutoff support; its cutoff and r^{-1} terms are smaller. Derivatives of the tilt variation cost $O(t^{-1/6})$ and $O(1)$, giving smaller errors after reordering. Derivatives of radial coefficients are bounded by their indicated powers of r^{-1}; since $\ell \ll r$, the listed cutoff bounds also dominate the mixed terms. Absorb the errors multiplying the localized state, square and sum. Bounded overlap controls the sum of the supported input norms independently of the number of intervals. Together with the first part this proves the proposition.

All inverse estimates are applied on the whole line or whole radial half-line to compactly supported states. Their commutators have been included. The computations first hold for smooth regular states and extend to $D(\mathcal{L})$ by its radial Dirichlet core; the reordered expressions involve only the original L^2 state and forcing. ▫

4. The exact six-component system [EXACT]

Use the flat physical variables $U = r^{1/2}(E, u)^T$, and define

$$Z = \begin{pmatrix} b\Psi/r^{5/2} \\ dp/r^{3/2} \end{pmatrix}, \qquad V = \frac{U'}{\sqrt{t}\, r}, \qquad X = (Z, U, V)^T, \qquad K = \sqrt{t}\, r^3, \qquad s = \operatorname{sgn}\tau.$$

Let

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_* = \operatorname{diag}(-1, 1), \quad S_* = \begin{pmatrix} 0 & 0 \\ -2\gamma/b & 0 \end{pmatrix}, \quad G_* = \operatorname{diag}(3/4, -1/4), \quad N_* = \operatorname{diag}(5/2, 3/2).$$

The original equations, without differentiating their forcing, are exactly

$$X' = r^{-2}A(K, r)X + r^{-1}B(K, r)X + \mathcal{F},$$

$$A(K, r) = \begin{pmatrix} 0 & bH & 0 \\ 0 & 0 & KI \\ (isK/v)(D_* + rS_*) & (is\gamma K/v)I & 0 \end{pmatrix}, \qquad B(K, r) = \begin{pmatrix} -N_* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & rK^{-1}G_* & -I \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} 0 \\ 0 \\ -F/(v\sqrt{t}\, r) \end{pmatrix}.$$

Each entry displayed as zero is a 2×2 block (or a two-vector for $\mathcal{F}$). In particular $|\mathcal{F}| \le Cr^2K^{-1}|F|$. The shear remains in rS_*.

5. An interval estimate between core and transport scales [EXACT]

Lemma 9.59. There are constants $K_0, r_0, \eta_0, C > 0$ such that on any interval $[a, c] \subset (0, r_0]$ satisfying

$$K(r) \ge K_0, \qquad K(r)^2 r = tr^7 \le \eta_0,$$

every actual six-component state satisfies

$$\int_a^c |X(r)|^2 dr \le C\left(a|X(a)|^2 + c|X(c)|^2 + \int_a^c r^7 |F(r)|^2 dr \right).$$

The constants are independent of the endpoints and t.

Proof: the six roots and their coordinates. The matrix

$$b(D_* + rS_*)H = \begin{pmatrix} 0 & -b \\ b & -2\gamma r \end{pmatrix}$$

has eigenvalues $\mu_{\pm} = -\gamma r \pm i\sqrt{b^2 - \gamma^2 r^2}$ for sufficiently small r, with uniformly independent eigenvectors $u_{\pm} = (1, -\mu_{\pm}/b)^T$. For each μ, the three eigenvalues of A solve

$$\nu\lambda^3 - is\gamma K^2\lambda - isK^2\mu = 0.$$

There is one slow root

$$\lambda_{\text{slow}} = -\mu/\gamma + \frac{\nu(-\mu/\gamma)^3}{is\gamma K^2} + O(K^{-4}),$$

and two fast roots $\lambda_{\text{fast},\pm} = \pm K\sqrt{is\gamma/\nu} + O(1)$. The estimates and their parameter derivatives are uniform for small r and K^{-1}. They follow directly from the implicit-function theorem, applied to the slow cubic after division by K^2, and to the fast cubic with $\lambda = K\ell$.

In particular the two slow real parts are

$$\text{Re}\lambda_{\text{slow},\pm} = r \pm \frac{s\nu b^3}{\gamma^4 K^2} + O(rK^{-2} + K^{-4}).$$

Choose K_0 large and η_0 small. Then, whenever $rK^2 \le \eta_0$, these have opposite signs and absolute values at least $c_0 K^{-2}$. The four fast roots have real parts of absolute value at least $c_0 K$, two of each sign.

An eigenvector for a root belonging to $u_{\pm}$ is

$$\left((b/\lambda)Hu_{\pm},\ u_{\pm},\ (\lambda/K)u_{\pm}\right)^T.$$

These six columns form a uniformly invertible matrix $T(K^{-1}, r)$, smooth with bounded first derivatives down to $K^{-1} = 0$. At that limit the four fast columns span the U, V coordinates, while the two slow columns have independent Z coordinates. The slow roots and their reciprocals are bounded away from zero; for the fast columns use $\lambda = K\ell$, so $b/\lambda = bK^{-1}/\ell$ and $\lambda/K = \ell$ are smooth. This also proves the asserted uniform invertibility rather than inferring it from distinct eigenvalues alone.

Put $Y = T^{-1}X$. Since $(K^{-1})' = -3K^{-1}/r$, its equation is

$$Y' = r^{-2}\Lambda Y + E(r)Y + T^{-1}\mathcal{F}, \qquad |E(r)| \le C/r.$$

Let Σ be the constant diagonal matrix of signs of the six real parts in this ordering. The eigenvalue bounds and $tr^7 \le \eta_0$ imply, after decreasing η_0 to absorb E and the derivative of the weight r,

$$\frac{d}{dr}\left(rY^*\Sigma Y\right) \ge \frac{c_1}{K^2 r}|Y|^2 - C_1 r^7|F|^2.$$

For the forcing term use $r|\mathcal{F}||Y|$ and Young's inequality: its squared remainder is bounded by $CK^2r^3|\mathcal{F}|^2 \le Cr^7|F|^2$. Finally $(K^2 r)^{-1} \ge \eta_0^{-1}$. Integrate and bound both endpoint quadratic forms by $Cr|X|^2$. This proves the estimate. ▫

This is an interval estimate for a single exact first-order system. The signature form is used only to estimate that system and its actual endpoint states; no height-mass symmetrizer or independently patched radial boundary problem is asserted.

6. Endpoint states and the contradiction [EXACT]

Theorem 9.60 (full frozen-window compactness). Suppose $|\tau_n| \to \infty$, $z_n \in D(\mathcal{L})$, and

$$\sup_n\left(\ \|\ z_n\ \|_{\mathcal{H}} + \|\ \mathcal{R}(\tau_n)z_n\ \|_{\mathcal{H}}\ \right) < \infty.$$

Then $\|\ z_n\ \|_{\mathcal{H}} \to 0$. In particular no normalized quasimode exists. Equivalently,

$$\inf_{\substack{z \in D(\mathcal{L}) \\ \| z \|_{\mathcal{H}} = 1}} \| \mathcal{R}(\tau) z \|_{\mathcal{H}} \to \infty \quad \text{as } |\tau| \to \infty.$$

Proof: inner endpoint control. Fix B large enough that $B^3 \geq K_0$. On any fixed core annulus $Bt^{-1/6} \leq r \leq 2Bt^{-1/6}$,

$$\int |X|^2 dr \to 0.$$

Here is the additional justification needed to pass from Proposition 9.58's U-norm to the full endpoint state. Cover the annulus by intervals of length $\ell = t^{-1/3}$, with enlarged intervals of bounded overlap within $[(B/2)t^{-1/6}, 3Bt^{-1/6}]$. In the coordinate $q = (r - r_j)/\ell$, the homogeneous coefficient matrix converges uniformly on each enlarged interval to

$$A_*(y) = y^{-2} A(y^3, 0), \qquad y = r_j t^{1/6} \in [B/2, 3B].$$

The error is $O_B(t^{-1/6})$, the family $A_*(y)$ is compact, and the rescaled forcing is bounded by $C_B t^{-2/3}|F|$.

Observation of U determines every homogeneous state in this finite-dimensional limiting family: if $U(q)$ vanishes on an interval, its equation forces $V = 0$, and the V equation then forces $D_* Z = 0$, hence $Z = 0$. The observation Gramian on a fixed interval is therefore positive definite. Its least eigenvalue has a positive minimum over the compact family $y \in [B/2, 3B]$ and the two signs s. Duhamel's formula and uniform convergence of the coefficient matrices preserve this bound for sufficiently large t. Thus on an interval and its enlargement,

$$\int_{I_j} |X|^2 dr \leq C_B \left(\int_{I_j^+} |U|^2 dr + t^{-4/3} \int_{I_j^+} |F|^2 dr \right).$$

This is a finite-dimensional observation estimate, with its injectivity and uniformity checked here. Summing by bounded overlap and using Proposition 9.58 proves the assertion. Choose $a_n \in [Bt_n^{-1/6}, 2Bt_n^{-1/6}]$ by averaging so that $a_n |X_n(a_n)|^2 \to 0$.

Outer endpoint control. Choose a fixed small $\delta > 0$ so that

$$t(2\delta h)^7 = \frac{\nu b^3}{7\gamma^4} (2\delta)^7 \leq \eta_0.$$

On $\delta h \leq r \leq 2\delta h$, Proposition 9.57 and its proved energy estimate give

$$\int |U_n|^2 dr \to 0, \qquad \int |V_n|^2 dr \to 0.$$

Indeed $\sqrt{h}\, U(hx) = v(x)$ and $\sqrt{h}\, V(hx)$ is a fixed constant times $h^{3/2} x^{-1} v'(x)$, controlled by the E_h term in that proposition. To control Z, write $Y_{\text{prim}} = \mathsf{F} C$ as in Attachment C. On each fixed rescaled annulus the Hankel bounds give $\sqrt{h}\, |Z(hx)| \leq C_\delta |C(hx)|/\sqrt{h}$. Moreover the exact correction gives

$$h^{-1/2} C(hx) = c(x) + h^2 A_h(x) v'(x) + h \tilde{A}_h(x) v(x),$$

with bounded matrices on this annulus. Proposition 9.56 gives $c \to 0$, and the derivative term has norm bounded by $C_\delta h^{1/2} (h^{3/2} \| v' \|_2) \to 0$. Thus $\int |Z_n|^2 dr \to 0$ as well. By averaging choose $c_n \in [\delta h_n, 2\delta h_n]$ with $c_n |X_n(c_n)|^2 \to 0$.

For large n, $a_n < c_n < r_0$, and all of Lemma 9.59's hypotheses hold on $[a_n, c_n]$. Its estimate and boundedness of F_n give

$$\int_{a_n}^{c_n} |U_n|^2 dr \leq \int_{a_n}^{c_n} |X_n|^2 dr \to 0,$$

since $\int_{a_n}^{c_n} r^7 |F_n|^2 dr \le c_n^7 \| F_n \|_2^2 \to 0$. Below a_n use Proposition 9.58 with $2B$; above c_n use Proposition 9.57 with δ. These three estimates exhaust the original physical norm, so $\| z_n \|_{\mathcal{H}}^2 = \int_0^R |U_n|^2 dr \to 0$.

If the final displayed infimum did not tend to infinity, a sequence of unit states with uniformly bounded residuals could be chosen, contradicting what was just proved. ▫

7. Scope and audit

This closes the frozen-window high-frequency compactness statement, including the former (C*) obstruction. It does not prove the numerical rate $|\tau|^{4/7}$. It is a lower-bound statement on the original Dirichlet domain; non-resonance on the finite-frequency interval not covered by Proposition 9.44, surjectivity, the source-fold term κt, height outflow, variable-profile transfer and the global inboard construction are separate statements.

The proof uses the original pressure endpoint through Proposition 9.56, actual reconstructed endpoint states in Lemma 9.59, and a norm estimate on every part of the radius interval. It never assumes that a transport-scale limit is nonzero. The endpoint observation estimate is proved independently on finite-dimensional systems, not inferred from the global compactness being established.

The script `check-inner-fold-closure.py` in the calculations repository [13] verifies the differentiated mass, radial diffusion, reordered cubic commutators and six-component equations symbolically, and checks the core observation rank. A numerical audit for both signs of γ and τ verifies the six eigenvectors and slow-root signs for $K = 100, 300, 1000, 3000$: the largest tested normalized eigenvector condition number is 4.376, and the largest relative matrix residual is below $8 \cdot 10^{-16}$. These checks audit the algebra; the uniform root and observation bounds are proved above. No numerical lower bound is used as a premise of Theorem 9.60.