Attachment C — Global Quasimodes of the Frozen Axis-Fold Operator

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Attachment C to Structural Classification of the Exterior-Exponent Collapse Mechanism (Paper I), 2026. It proves Attachment A, Propositions 9.51–9.52, Lemma 9.53 and Propositions 9.54–9.57; full frozen-window compactness is completed by Theorem 9.60 in Attachment D. "Frozen" refers to the axis-fold operator with its coefficients fixed. Bracketed citations refer to the reference list of Paper I; labels such as cv30 identify runs whose scripts are in the calculations repository [13].

1. The operator and the scope [DEF / CLASS]

Fix $R, \nu, d, \kappa_E > 0$ and $\gamma \neq 0$. Write

$$\mathcal{H} = L^2((0,R), \tfrac{\kappa_E}{4} r^3 dr) \oplus L^2((0,R), r\,dr), \qquad \mathcal{L} = -\nu \mathrm{diag}(\Delta_4, \Delta_2),$$

with homogeneous Dirichlet conditions at R and the regular radial realization at the axis. Here $\nu = 2L$ in Paper I. Define

$$\Psi(r) = \int_0^r \rho u(\rho) d\rho, \quad \mathfrak{M}u = \frac{2\Psi}{r^2}, \quad p(r) = -\frac{\kappa_E}{2} \int_r^R \rho e(\rho) d\rho,$$

$$\mathcal{A}_0(e,u) = \Big(\gamma r^2 e - d\mathfrak{M}u,\ \gamma r^2 u - 2\gamma\Psi + dp \Big), \qquad \mathcal{R}(\tau) = \mathcal{L} + i\tau\mathcal{A}_0.$$

These are the full frozen-window operators used in the earlier computations, including the shear and both integration endpoints. The term κt, variable profile coefficients, the height ends, and global inboard patching are outside this frozen problem.

The target is to exclude $|\tau_n| \to \infty$, $\| z_n \|_{\mathcal{H}} = 1, f_n := \mathcal{R}(\tau_n) z_n \to 0$ in $\mathcal{H}$. Throughout, exact statements about these operators are distinguished from balances and from estimates for proposed limit models.

2. Global concentration at the axis [EXACT]

Write $\mathcal{A}_0 = \gamma r^2 I + K$. The operator K is compact on $\mathcal{H}$. For example, under $E = (\sqrt{\kappa_E}/2) re$, which puts both components in $L^2(r dr)$, the two pair kernels relative to $\rho d\rho$ are

$$-\frac{b}{r}\mathbf{1}_{\rho < r}, \qquad -\frac{b}{\rho}\mathbf{1}_{r < \rho}, \qquad b = d\sqrt{\kappa_E}.$$

Each has squared Hilbert–Schmidt norm $b^2 R^2/4$. The full shear kernel is $-2\gamma \mathbf{1}_{\rho < r}$, with squared Hilbert–Schmidt norm $\gamma^2 R^4/2$. No endpoint has been dropped.

Proposition 9.51. Every hypothetical quasimode sequence above satisfies

$$z_n \rightharpoonup 0, \qquad \| K z_n \|_{\mathcal{H}} \to 0, \qquad \int r^2 |z_n|_{\mathcal{H}}^2 \to 0,$$

$$\langle \mathcal{L} z_n, z_n \rangle \to \infty, \qquad \frac{\langle \mathcal{L} z_n, z_n \rangle}{|\tau_n|} \to 0, \qquad \frac{\| \mathcal{L} z_n \|_{\mathcal{H}}}{|\tau_n|} \to 0.$$

In particular $\| \mathbf{1}_{r \geq \delta} z_n \|_{\mathcal{H}} \to 0$ for every fixed $\delta > 0$.

Proof. If z is any weak subsequential limit, testing $\mathcal{R}(\tau_n) z_n = f_n$ against $\phi \in D(\mathcal{L})$ gives

$$\langle \mathcal{A}_0 z_n, \phi \rangle = \frac{\langle f_n, \phi \rangle - \langle z_n, \mathcal{L}\phi \rangle}{i\tau_n} \to 0.$$

Boundedness of $\mathcal{A}_0$ and Lemma 9.48 imply $\mathcal{A}_0 z = 0$, hence $z = 0$. Compactness gives $Kz_n \to 0$ strongly. Taking the imaginary part of the full resolvent pairing gives

$$\gamma \int r^2 |z_n|_{\mathcal{H}}^2 = \frac{\mathrm{Im}\langle f_n, z_n\rangle}{\tau_n} - \mathrm{Re}\langle Kz_n, z_n\rangle \to 0.$$

Consequently $r^2 z_n \to 0$ strongly, since $\| r^2 z_n \|^2 \le R^2 \int r^2 |z_n|_{\mathcal{H}}^2$, and therefore $\mathcal{A}_0 z_n \to 0$ strongly. The equation gives the last asserted limit. Taking real parts and using compactness of K gives the energy divided by $|\tau_n|$ tending to zero. A bounded-energy subsequence would converge strongly by $D(\mathcal{L}^{1/2}) \Subset \mathcal{H}$ and contradict its unit norm and weak limit zero. ▫

This is global spatial concentration, not a rate of concentration or control of boundary derivatives. In particular it does not give $|\tau_n| \| Kz_n \| \to 0$.

A second global identity [EXACT]. For smooth regular $z = (e, u)$ with $e(R) = u(R) = 0$, put $Wz = (re_r + 3e, ru_r + u)$. Direct integration by parts gives

$$\mathrm{Re}\langle \mathcal{A}_0 z, Wz\rangle_{\mathcal{H}} = \gamma \left(\int_0^R r^3 |u|^2 dr + |\Psi(R)|^2 \right).$$

The two pair terms have opposite real parts; the e-tilt contribution is zero. In the u-row the tilt contributes $-\gamma \int r^3 |u|^2$, while the shear contributes $2\gamma \int r^3 |u|^2 + \gamma |\Psi(R)|^2$. The pressure endpoint is kept in this cancellation. For $\gamma > 0$ this is a positive global virial of the full mass, including an outer primitive term. Applied to the resolvent it gives

$$\tau\gamma \left(\int r^3 |u|^2 + |\Psi(R)|^2 \right) = \mathrm{Im}\langle f, Wz\rangle - \mathrm{Im}\langle \mathcal{L}z, Wz\rangle.$$

It is not yet a coercive resolvent estimate: W differentiates the state, and the last term and boundary-layer derivatives require control. It supplies an exact global identity for that task without a localized symmetrizer.

3. A test of the proposed local transfer [BAL / method limitation]

At a mode-zero packet of radius ρ, the characteristic relation gives $|\xi_0| \asymp \rho^{-2}$. Comparability with the transverse Airy frequency $(|\tau|\rho)^{1/3}$ therefore gives

$$|\tau| \asymp \rho^{-7}, \qquad q \asymp \rho^{-4} \asymp |\tau|^{4/7}.$$

For a spatial cutoff of width w, the principal commutator of the order-minus-one pair has size $b/(\xi_0^2 w)$. Its resolvent contribution divided by q is of size

$$\frac{|\tau| b}{q \xi_0^2 w} \asymp \frac{\rho}{w}.$$

The quadratic remainder in freezing γr^2 to its tangent has relative size

$$\frac{|\tau|\,|\gamma| w^2}{q} \asymp \frac{w^2}{\rho^3}.$$

Uniform absorption by these norm estimates would require both $w \gg \rho$ and $w \ll \rho^{3/2}$, which have no overlap as $\rho \to 0$.

This rules out that particular cutoff-and-freezing estimate in the comparable-frequency regime. It does not prove that every localization is impossible, nor construct a quasimode. It explains why the whole-line lower bound of Lemma 9.50 cannot simply be assigned to a fine partition of the window. A global propagation argument must retain the motion along the characteristic curve.

4. The pressure endpoint carries a reflection relation [EXACT for the inviscid mass equation]

Proposition 9.52. Consider $\mathcal{A}_0 z = 0$ away from the axis, retaining $p(R) = 0$ but temporarily allowing nonzero inviscid e, u traces at R. With $\beta = b/|\gamma|$, the general primitive is

$$\Psi = \frac{r}{\beta}\Big(AH_1^{(1)}(\beta/r) + BH_1^{(2)}(\beta/r) \Big).$$

The differentiated axial row is $\gamma r^2 u_r + dp_r = 0$, and its undifferentiated boundary value imposes

$$R\Psi'(R) - 2\Psi(R) = 0.$$

Using $H_1'(y) + H_1(y)/y = H_0(y)$ yields

$$B = -\frac{H_0^{(1)}(\beta/R)}{H_0^{(2)}(\beta/R)}A, \qquad |B| = |A|.$$

The denominator is nonzero on the positive real axis, since J_0 and Y_0 do not vanish together. Thus the pressure endpoint supplies a unit-modulus relation between the two oscillatory coefficients. The derivative recurrence and nonvanishing Wronskian used here are recorded in DLMF §10.6 and DLMF §10.5.

This calculation deliberately retains the integration constant lost by differentiating the axial row. Proposition 9.54 below transfers the relation to actual Dirichlet resolvent states on an outer collar whose inner radius tends to zero, and Proposition 9.56 passes it to the transport limit. If the inviscid Dirichlet conditions were also imposed directly, the available outer solutions would be removed rather than matched.

5. Transport on the full curve [BAL; its boundary-value estimate is EXACT]

Take $\gamma, \tau > 0$ for the displayed balance; the other signs exchange the two directions. In flat radial $L^2(dr)$ modal coordinates, the mass-zero phases have $\phi_\pm' = \pm\beta/r^2$. Their leading amplitudes satisfy the transport balances

$$\mathcal{T}_\pm a_\pm = \pm\tau\frac{\gamma^2}{b}(r^4 a_\pm' + r^3 a_\pm) + \nu\beta^2 r^{-4} a_\pm.$$

The coefficient $r^3 a$ includes the shear. It is consistent with the exact Bessel mass-kernel amplitude $a \asymp r^{-1}$; dropping the shear changes that amplitude. This is an expansion along the curve, not an asserted operator-norm equivalence for arbitrary states.

At the retained symbolic order the two group velocities are $v_\pm = \pm\gamma^2 r^4/b$, and the projected imaginary mass terms are $\iota_\pm = \pm\gamma^2 r^3/b$. The amplitude equation contains $\tau[v_\pm a' + (v_\pm'/2 - \iota_\pm)a] + \nu(\phi_\pm')^2 a$, giving the displayed coefficient. Controlling all discarded terms and the fast modes is part of the reduction still to be proved.

Put

$$\rho_\tau = \left(\frac{\nu b^3}{7\gamma^4\tau}\right)^{1/7}, \quad s = (\rho_\tau/r)^7, \quad a_\pm = r^{-1}B_\pm(s), \quad q_\tau = \nu\beta^2\rho_\tau^{-4}.$$

Direct substitution gives

$$\mathcal{T}_\pm a_\pm = r^{-1} q_\tau s^{4/7}(\mp B_\pm' + B_\pm), \qquad \| a_\pm \|_{L^2(dr)}^2 = \frac{1}{7\rho_\tau}\int s^{-6/7}|B_\pm(s)|^2 ds.$$

The fixed outer endpoint corresponds to $s_R = (\rho_\tau/R)^7 \to 0$. Its reflection relation must be retained at $s = 0$ in a proposed limiting problem.

Lemma 9.53 (a uniform estimate for the reflected transport model). Let $\mathcal{K} = L^2((0,\infty), s^{-6/7}ds)$ and $|\zeta| = 1$. On locally absolutely continuous pairs with the displayed operator in $\mathcal{K}^2$, impose

$$B_-(0) = \zeta B_+(0), \qquad \mathcal{T}(B_+, B_-) = \left(s^{4/7}(-B_+' + B_+), s^{4/7}(B_-' + B_-)\right).$$

There is a bounded inverse $\mathcal{K}^2 \to \mathcal{K}^2$, with bound independent of ζ. In particular $\| B \|_{\mathcal{K}^2} \le C \| \mathcal{T}B \|_{\mathcal{K}^2}$ uniformly in the reflection phase.

Proof. For forcing (F_+, F_-), the only possible weighted-square-integrable outgoing component is

$$B_+(s) = e^s \int_s^\infty e^{-t} t^{-4/7} F_+(t) dt.$$

Its trace exists and obeys

$$|B_+(0)| \le \left(\int_0^\infty e^{-2t} t^{-2/7} dt \right)^{1/2} \| F_+ \|_{\mathcal{K}} .$$

The incoming component is necessarily

$$B_-(s) = \zeta B_+(0) e^{-s} + \int_0^s e^{-(s-t)} t^{-4/7} F_-(t) dt.$$

These formulas give uniqueness and the boundary condition. For completeness, their norm bounds follow without an unproved weighted resolvent theorem. On $s < 1$, B_+ is uniformly bounded by the trace estimate with an extra factor e; $s^{-6/7}$ is integrable there. The integral term of B_- obeys

$$\left| \int_0^s e^{-(s-t)} t^{-4/7} F_-(t) dt \right|^2 \le \frac{7}{5} s^{5/7} \| F_- \|_{\mathcal{K}}^2 ,$$

whose weighted square norm on $(0,1)$ is at most $49/30$ times the forcing norm squared. On $s, t \ge 1$, conjugation by the square root of the weight gives the kernel factor $s^{-3/7} t^{-1/7} \le 1$; the remaining one-sided exponential convolution has L^2 norm at most one. The part $t < 1 < s$ decays exponentially and is bounded by the same Cauchy–Schwarz argument. Finally $e^{-s} \in \mathcal{K}$. All constants are independent of the phase ζ. Local absolute continuity follows from the formulas, and the differential equations hold almost everywhere. Conversely the equations make both traces exist, since $s^{-4/7} F \in L^1(0,1)$. The excluded homogeneous outgoing term is e^s, and the boundary relation fixes the remaining e^{-s} term. ▫

The countertest. If the reflection relation is omitted, $(B_+, B_-) = (0, e^{-s})$ is a nonzero kernel in $\mathcal{K}^2$. It corresponds to $a_- = r^{-1}\exp[-(\rho_\tau/r)^7]$. Thus square-integrable local limits alone do not discharge this global problem. The outer trace can survive rescaling as a relation at infinity even while the original $\mathcal{H}$ mass concentrates at the axis. This is a domain distinction, not a counterexample to the original resolvent.

6. Full-window numerical check [NUM]

The script `check-global-fold-quasimodes.py` in the calculations repository [13] realizes the two primitives as auxiliary sparse equations. It uses the complete frozen mass and the same radial finite-volume discretization as cv30; it makes no spatial cutoffs. An independent comparison with cv30 at 60 nodes and $\tau = 10^4$ gave relative operator error $2.2 \cdot 10^{-16}$ and inverse error $5.3 \cdot 10^{-15}$. Both forward and adjoint singular-vector equations are checked.

For $R = 1, \nu = 2, \gamma = 0.5, d = 0.9, \kappa_E = 1$:

τ	$\sigma_{\min}$, 4,000 nodes	$\sigma_{\min}$, 8,000 nodes	8,000-node $\sigma_{\min}/\tau^{4/7}$	8,000-node median radius times $\tau^{1/7}$
10^6	3071.439	3071.495	1.14493	2.49776

τ	$\sigma_{\min}$, 4,000 nodes	$\sigma_{\min}$, 8,000 nodes	8,000-node $\sigma_{\min}/\tau^{4/7}$	8,000-node median radius times $\tau^{1/7}$
10^8	41756.217	41759.634	1.12028	2.73470
10^{10}	574003.641	574353.854	1.10890	2.78833

The two finest resolutions differ by at most 0.061% in this table. Forward relative residuals are below $3 \cdot 10^{-9}$ and adjoint residuals below $4 \cdot 10^{-12}$. This supports the $4/7$ balance and supplies no quasimode. It is neither a lower-bound proof nor verification of the limit boundary relation. The finite-volume boundary convention is inherited from cv30 and converges under refinement; the tabulated values are not exact continuum singular values.

7. Outer reflection for actual viscous states [EXACT]

The primitive variables retain the pressure trace even inside a viscous boundary layer. Put $Y = (\Psi, p)^T$ and $g = \mathcal{A}_0 z$, ordered as (g_e, g_u). Directly from the two mass rows,

$$Y' = \mathsf{B}Y + \mathsf{D}g, \qquad \mathsf{B} = \begin{pmatrix} 2/r & -d/(\gamma r) \\ d\kappa_E/(\gamma r^3) & 0 \end{pmatrix}, \qquad \mathsf{D} = \begin{pmatrix} 0 & 1/(\gamma r) \\ \kappa_E/(2\gamma r) & 0 \end{pmatrix}.$$

Let F have the two columns

$$Y_j(r) = \begin{pmatrix} (r/\beta)H_1^{(j)}(\beta/r) \\ (\gamma/d)H_0^{(j)}(\beta/r) \end{pmatrix}, \qquad j = 1, 2.$$

Then $\mathsf{F}' = \mathsf{B}\mathsf{F}$ and

$$\det\mathsf{F} = -\frac{4i\gamma r^2}{\pi d\beta^2} \neq 0.$$

Define the **actual primitive coefficients** $C = \mathsf{F}^{-1}Y$ and $\mathsf{M} = \mathsf{F}^{-1}\mathsf{D}$. These are defined for every $z \in D(\mathcal{L})$ on every closed collar away from the axis, without an inviscid approximation. They obey

$$C' = \mathsf{M}g, \qquad C_2(R) = \zeta C_1(R), \qquad \zeta = -H_0^{(1)}(\beta/R)/H_0^{(2)}(\beta/R).$$

The boundary relation is exact because $p(R) = 0$ in the original operator. It imposes no new trace on e, u.

Proposition 9.54 (quantitative outer reflection transfer). Fix the parameters of §1, and choose $0 < a_0 \le \min(1, R/2)$. There is a constant C_* depending only on these fixed parameters and a_0 such that, for $t = |\tau| \ge 1$, $z \in D(\mathcal{L})$, $f = \mathcal{R}(\tau)z$, and $t^{-1/2} \le a \le a_0$,

$$\sup_{a \le r \le R} |C(r) - C(R)| \le C_* \left(t^{-1/4}a^{-3} + t^{-3/4}a^{-5} + t^{-1}a^{-13/2} \right) \left(\| z \|_{\mathcal{H}} + \| f \|_{\mathcal{H}} \right).$$

In particular, with $a_t = a_0 t^{-1/28}$ and $\rho_t = (\nu b^3/(7\gamma^4 t))^{1/7}$, for all sufficiently large t,

$$\sup_{a_t \le r \le R} \frac{|C_2(r) - \zeta C_1(r)|}{\sqrt{\rho_t}} \le C_* t^{-1/14} \left(\| z \|_{\mathcal{H}} + \| f \|_{\mathcal{H}} \right).$$

The estimate is valid for either sign of γ or τ. For normalized quasimodes its right-hand side tends to zero.

Proof. Write $N_z = \| z \|_{\mathcal{H}} + \| f \|_{\mathcal{H}}$. Boundedness of $\mathcal{A}_0$ and positivity of $\mathcal{L}$ give

$$\| \mathcal{L}z \|_{\mathcal{H}} \le CtN_z, \qquad \| \mathcal{L}^{1/2}z \|_{\mathcal{H}} \le Ct^{1/2}N_z.$$

On $[a, R]$ the component weights therefore imply

$$\begin{aligned} \| e \|_2 &\le Ca^{-3/2}N_z, & \| u \|_2 &\le Ca^{-1/2}N_z,\\ \| e' \|_2 &\le Ca^{-3/2}t^{1/2}N_z, & \| u' \|_2 &\le Ca^{-1/2}t^{1/2}N_z,\\ \| e'' \|_2 &\le Ca^{-3/2}tN_z, & \| u'' \|_2 &\le Ca^{-1/2}tN_z. \end{aligned}$$

For the last line use $e'' = \Delta_4 e - 3e'/r$ and $u'' = \Delta_2 u - u'/r$, with $a^{-1} \le t^{1/2}$. All these unweighted norms are on the collar. The elementary one-dimensional bound $\| v \|_\infty^2 \le C(\| v \|_2^2 + \| v \|_2 \| v' \|_2)$ has a uniform constant here because $R - a \ge R/2$. Applied to e, u and then to e', u', it gives

$$\begin{aligned} \| e \|_\infty &\le Ca^{-3/2}t^{1/4}N_z, & \| u \|_\infty &\le Ca^{-1/2}t^{1/4}N_z,\\ \| e' \|_\infty &\le Ca^{-3/2}t^{3/4}N_z, & \| u' \|_\infty &\le Ca^{-1/2}t^{3/4}N_z. \end{aligned}$$

The explicit inverse of F, its nonzero determinant, and the fixed-order large-positive-argument Hankel bounds give, for $j = 0, 1, 2$,

$$|\partial_r^j \mathsf{M}_{\cdot e}(r)| \le Cr^{-3/2-2j}, \qquad |\partial_r^j \mathsf{M}_{\cdot u}(r)| \le Cr^{-5/2-2j}, \qquad 0 < r \le R.$$

Here the first bounds follow from DLMF §10.17; differentiation uses the Hankel recurrences and their differential equation. Constants extend over the remaining compact interval by smoothness. The determinant uses DLMF §10.5.

Set $\mathsf{N} = \mathrm{diag}(3, 1)$ and $\mathsf{Q} = \mathsf{MN}/r - \mathsf{M}'$. Since $g = (f + \nu \mathrm{diag}(\Delta_4, \Delta_2)z)/(i\tau)$, integration by parts gives the exact formula

$$C(R) - C(r) = \frac{1}{i\tau}\int_r^R \mathsf{M} f \, d\ell + \frac{\nu}{i\tau}\Big[\mathsf{M}z' + \mathsf{Q}z\Big]_r^R - \frac{\nu}{i\tau}\int_r^R \mathsf{Q}' z \, d\ell.$$

The two boundary terms are bounded by $Ca^{-3}t^{3/4}N_z$ and $Ca^{-5}t^{1/4}N_z$. For the integrals, weighted Cauchy–Schwarz gives

$$\int_a^R |\mathsf{M} f| \, dr \le Ca^{-5/2} \| f \|_{\mathcal{H}}, \qquad \int_a^R |\mathsf{Q}' z| \, dr \le Ca^{-13/2} \| z \|_{\mathcal{H}}.$$

Indeed, after dividing either column by its component weight's square root, the relevant pointwise bounds are Cr^{-3} and Cr^{-7}, respectively. Multiplication by t^{-1} proves the first assertion, absorbing the forcing term since $a \le 1$. The argument is valid on the radial Dirichlet operator domain: its elements are H^2 on these collars, so the derivative traces and integrations by parts used above exist.

At $a = a_t$, the three powers in the bound are $t^{-1/7}$, $t^{-4/7}$ and $t^{-43/56}$. Dividing by $\sqrt{\rho_t} \asymp t^{-1/14}$ leaves a leading $t^{-1/14}$. Finally use $C_2(R) = \zeta C_1(R)$ and $|\zeta| = 1$. ▫

This closes transmission across the outer viscous boundary layer **in primitive coefficients**. This estimate alone does not establish a transport trace at $s = 0$: substituting $a \asymp \rho_t$ does not give a vanishing normalized error. Proposition 9.56 below supplies the missing propagation by integrating the corrected equation directly to the original outer endpoint.

8. Exact bulk identities and the norm that must survive [EXACT / OPEN]

The integration formula also supplies corrected coefficients

$$\tilde{C} = C - \frac{\nu}{i\tau}(\mathsf{M}z' + \mathsf{Q}z), \qquad \tilde{C}' = \frac{1}{i\tau}\mathsf{M} f - \frac{\nu}{i\tau}\mathsf{Q}' z.$$

This is an identity on each collar, not the transport approximation. Its right-hand side has no derivatives of f or z. In particular $\tilde{C}$ is locally absolutely continuous for $z \in D(\mathcal{L})$. No assertion that $\tilde{C} - C$ is small at the transport scale follows just from this formula.

There is a separate, exact reconstruction test. Define

$$\mathsf{L}_0 = \begin{pmatrix} \gamma r^4/(2d) & 0 \\ \gamma^2 r^4/d^2 & -\gamma r^2/d \end{pmatrix}, \qquad \mathsf{L}_1 = \begin{pmatrix} -r^2/(2d) & 0 \\ -\gamma r^2/d^2 & 1/d \end{pmatrix}.$$

The two mass rows give $Y = \mathsf{L}_0 z + \mathsf{L}_1 g$, with $\mathsf{L}_0^{-1}\mathsf{L}_1 = -(\gamma r^2)^{-1} I$. Thus, with $\mathsf{W} = \mathsf{L}_0^{-1}\mathsf{F}$,

$$\boxed{z = \mathsf{W}C + \frac{\mathcal{A}_0 z}{\gamma r^2}.}$$

Put $\mathsf{H}(r) = \mathrm{diag}(\kappa_E r^3/4, r)$. The leading Hankel terms yield the matrix estimate

$$r^2 \mathsf{W}(r)^* \mathsf{H}(r) \mathsf{W}(r) = \frac{4}{\pi\beta} I + O(r), \qquad r \downarrow 0.$$

The leading off-diagonal terms cancel between the e and u components; this is a pointwise matrix estimate, without any assumption that C varies slowly. Consequently, for a sufficiently small fixed r_0,

$$c \int_I r^{-2} |C|^2 dr \le \| \mathbf{1}_I \mathsf{W} C \|_{\mathcal{H}}^2 \le C \int_I r^{-2} |C|^2 dr, \qquad I \subset (0, r_0).$$

This explains the $\sqrt{\rho_t}$ normalization on $r \asymp \rho_t$. It also exposes the exact missing reconstruction estimate: $\mathcal{A}_0 z_n \to 0$ in $\mathcal{H}$ from Proposition 9.51 does **not** imply $r^{-2}\mathcal{A}_0 z_n \to 0$ there. Retention of physical norm in C_n requires control of this weighted defect (or an alternative reconstruction using $\tilde{C}_n$), as well as tightness on the proposed scale. Convergence of primitive coefficients alone cannot replace either obligation.

The script `check-primitive-reflection.py` in the calculations repository [13] checks the primitive and reconstruction identities symbolically, the Hankel determinant, the leading transport diagonalization, the full differential reconstruction and its flat-coordinate matrices, and the integrated corrected-coefficient identity on smooth complex Dirichlet states for both signs of γ and τ. These are sign and endpoint audits, not numerical proofs of Proposition 9.54's bounds.

9. Compactness of corrected amplitudes on rescaled annuli [EXACT]

Proposition 9.55. Let $t_n = |\tau_n| \to \infty$, $z_n \in D(\mathcal{L})$, $f_n = \mathcal{R}(\tau_n) z_n$, and suppose $\| z_n \|_{\mathcal{H}} + \| f_n \|_{\mathcal{H}} \le N$. Put $h_n = \rho_{t_n}$ and

$$c_n(x) = h_n^{-1/2} \tilde{C}_n(h_n x), \qquad 0 < x < R/h_n.$$

For each compact interval $I \subset (0, \infty)$ of positive length, $\| c_n \|_{H^1(I)} \le C_I N$ for all sufficiently large n. A subsequence therefore converges locally uniformly on $(0, \infty)$ and weakly in H^1 on compact intervals. Choose its sign $\sigma = \mathrm{sgn}(\gamma \tau_n)$ to be constant. Its limit satisfies

$$c' = 7\sigma x^{-8} \mathrm{diag}(1, -1) c, \qquad \int_0^\infty x^{-2} |c(x)|^2 dx \le \frac{\pi\beta}{4} \liminf_n \| z_n \|_{\mathcal{H}}^2 .$$

The liminf here and below is along the chosen subsequence. For $\sigma = 1$, necessarily $c = (Ae^{-x^{-7}}, 0)$; for $\sigma = -1$, necessarily $c = (0, Ae^{-x^{-7}})$. This assertion applies to hypothetical quasimodes but does not assert that $A \neq 0$.

Proof. Suppress n and define the normalized state and forcing by

$$v(x) = \begin{pmatrix} (\sqrt{\kappa_E}/2) h^2 x^{3/2} e(hx) \\ h x^{1/2} u(hx) \end{pmatrix}, \qquad w(x) = \begin{pmatrix} (\sqrt{\kappa_E}/2) h^2 x^{3/2} f_e(hx) \\ h x^{1/2} f_u(hx) \end{pmatrix}.$$

The change of variables is isometric: $\| v \|_{L^2(0, R/h)} = \| z \|_{\mathcal{H}}$ and $\| w \|_2 = \| f \|_{\mathcal{H}}$. By §8,

$$c'(x) = \frac{h^{1/2}}{i\tau} \Big(\mathsf{M}(hx) f(hx) - v \mathsf{Q}'(hx) z(hx) \Big).$$

On I, the two terms in parentheses have bounds $C_I h^{-7/2}|w|$ and $C_I h^{-15/2}|v|$. Since $|\tau|^{-1} = Ch^7$, this proves

$$\| c' \|_{L^2(I)} \le C_I(\| v \|_{L^2(I)} + h^4 \| w \|_{L^2(I)}) \le C_I N.$$

It remains to bound the additive constant; a derivative bound alone is insufficient.

Use the exact algebraic expression from §8,

$$C = \mathsf{F}^{-1}\mathsf{L}_0\left(z - \frac{f + \nu \mathrm{diag}(\Delta_4, \Delta_2) z}{i\tau\gamma r^2} \right),$$

and then subtract the correction defining $\tilde{C}$. Substitution of $r = hx$ and of the normalized state writes the result exactly, on I, as

$$c = A_{0,h} v + h A_{1,h} v + h^2 A_{2,h} v' + h^3 A_{3,h} v'' + h^5 A_{4,h} w.$$

The matrices here are obtained by collecting the zeroth, first and second derivatives of v; their dependence on $\mathrm{sgn}\tau$ is bounded. They satisfy

$$\| A_{j,h} \|_{L^\infty(I)} \le C_I \quad (0 \le j \le 4), \qquad \| A'_{2,h} \|_\infty \le C_I h^{-1}, \qquad \| A^{(k)}_{3,h} \|_\infty \le C_I h^{-k} \quad (k = 1, 2).$$

To check the powers directly, the columns of $\mathsf{F}^{-1}\mathsf{L}_0$ have sizes $r^{5/2}$ and $r^{3/2}$. Multiplying by $h^{-1/2} z(hx)$ gives order one. The Laplacian term adds $|\tau|^{-1} r^{-2} h^{-2} = O(h^3)$; its lower derivatives have the same or smaller bounds. The correction $h^{-1/2}\nu \mathsf{M} z'/(i\tau)$ has first-derivative coefficient $O(h^2)$, and $h^{-1/2}\nu \mathsf{Q} z/(i\tau)$ has zeroth-order coefficient $O(h)$. Differentiating the Hankel factors $H_j^{(k)}(\beta/(hx))$ with respect to x costs at most h^{-1} on I, which gives precisely the displayed derivative bounds. No bound on v' or v'' has been assumed.

Pair this formula with a fixed smooth function ϕ compactly supported in the interior of I and with $\int_I \phi = 1$. Integrate the v' and v'' terms by parts once and twice. Their bounds are $C_I h N \| \phi \|_{H^2(I)}$; all other terms are bounded by $C_I N \| \phi \|_{H^2(I)}$. Hence $| \int_I \phi c | \le C_I N$. For every $x \in I$,

$$\left| c(x) - \int_I \phi(y) c(y) dy \right| \le |I|^{1/2} \| \phi \|_{L^1(I)} \| c' \|_{L^2(I)}.$$

This proves the claimed H^1 bound. The same derivative estimate gives uniform $1/2$-Hölder continuity, so a diagonal subsequence on nested compact intervals converges locally uniformly; weak H^1 convergence follows from boundedness.

To identify the equation, let $T_h(x) = \mathrm{diag}(2h^{-2}x^{-3/2}/\sqrt{\kappa_E}, h^{-1}x^{-1/2})$, so $z(hx) = T_h v$. The leading matrix in the preceding expression is $A_{0,h} = h^{-1/2}\mathsf{F}(hx)^{-1}\mathsf{L}_0(hx)T_h$. It and its inverse are uniformly bounded on I, with their first two derivatives bounded by $C_I h^{-j}$. The exact corrected equation is $c' = D_h v + O_{L^2(I)}(h^4 \| w \|_2)$, where

$$D_h = -\frac{\nu h^{1/2}}{i\tau}\mathsf{Q}'(hx)T_h, \qquad D_h A_{0,h}^{-1} = -\frac{\nu h}{i\tau}\mathsf{Q}'(hx)\mathsf{W}(hx) =: \mathsf{R}_h(x).$$

Multiplying the exact expression for c by R_h and integrating its derivative terms by parts against any fixed smooth interior test function gives

$$c' - \mathsf{R}_h c \to 0 \quad \text{in distributions on } I.$$

Indeed, the remainder terms have the same sizes h, h^2, h^3 as before; the first two derivatives of R_h cost at most h^{-j}. Thus each differentiated-state contribution after integration by parts is $O(h)N$ in the test-function bound.

For clarity the leading matrix calculation is explicit. Put $\mathsf{J} = \mathsf{D}\mathsf{N}/r - \mathsf{D}' + \mathsf{B}\mathsf{D}$ and $\mathsf{E} = \mathsf{J}' - \mathsf{B}\mathsf{J}$, so $\mathsf{Q}' = \mathsf{F}^{-1}\mathsf{E}$. Direct multiplication gives

$$\mathrm{EL}_0^{-1} = \begin{pmatrix} 10d^2\kappa_E/(\gamma^3 r^7) - 32/(\gamma r^5) & d(-d^2\kappa_E + 16\gamma^2 r^2)/(\gamma^4 r^7) \\ d\kappa_E(d^2\kappa_E - 24\gamma^2 r^2)/(\gamma^4 r^9) & 8d^2\kappa_E/(\gamma^3 r^7) \end{pmatrix}.$$

The leading Hankel columns diagonalize its r^{-8} order after scaling the first primitive by r^{-1}. Specifically,

$$\mathrm{Q}'\mathrm{W} = \frac{b^3}{\gamma^4 r^8}\mathrm{diag}(-i\mathrm{sgn}\gamma, i\mathrm{sgn}\gamma) + O(r^{-7}).$$

The oscillatory off-diagonal factors have modulus one and occur only in the remainder. Since $h^7 = \nu b^3/(7\gamma^4|\tau|)$, this proves

$$\mathrm{R}_h(x) = 7\sigma x^{-8}\mathrm{diag}(1,-1) + O_I(h).$$

Local uniform convergence now identifies the asserted differential equation without assuming strong reconstruction of z_n.

Finally the earlier integrations by parts also give $c_n - A_{0,h_n}v_n \to 0$ in distributions. The norm matrix in §8 yields, uniformly on I,

$$A_{0,h}A_{0,h}^{*} = \frac{\pi\beta}{4}x^2 I + O_I(h).$$

Thus $x^{-1}A_{0,h_n}v_n$ is bounded in $L^2(I)$ and has weak limit $x^{-1}c$. Lower semicontinuity gives the asserted norm bound first on I, and then on $(0,\infty)$ by exhaustion. Solving the diagonal differential equations gives $c_1 = Ae^{-\sigma x^{-7}}$ and $c_2 = Be^{\sigma x^{-7}}$. Weighted square integrability at $x = 0$ removes the growing component. ▫

Scope of this compactness. This identifies actual local limits with the square-integrable branch of the homogeneous transport model. Under $s = x^{-7}$ the weight $x^{-2}dx$ becomes $\frac{1}{7}s^{-6/7}ds$; the component ordering is exchanged when needed to use Lemma 9.53, with a unit-modulus reflection phase in either ordering. The outer reflection does not follow from local convergence alone; Proposition 9.56 proves it using a uniform derivative tail bound. The upper norm bound here is not a lower bound retaining physical mass.

10. The outer reflection survives in the limit [EXACT]

Proposition 9.56 (reflection and vanishing of the transport limits). Under the hypotheses and notation of Proposition 9.55, for $1 \le L \le R/h_n$,

$$|c_{n,2}(L) - \zeta c_{n,1}(L)| \le CN\left(L^{-13/2} + h_n^4 L^{-5/2} + h_n^{5/4}\right).$$

Consequently every limit in Proposition 9.55 is zero. In fact the whole sequence converges locally uniformly to zero, $c_n(R/h_n) \to 0$, and for every $\varepsilon > 0$,

$$\int_{\varepsilon}^{R/h_n} x^{-2}|c_n(x)|^2 dx \to 0.$$

Proof. The column bounds in §7 are valid on the entire interval $0 < r \le R$, with constants depending on R. Applying them to the exact corrected equation and the normalized state of §9 gives the global pointwise estimate

$$|c_n'(x)| \le C\left(x^{-7}|v_n(x)| + h_n^4 x^{-3}|w_n(x)|\right), \qquad 0 < x < R/h_n.$$

Thus for $L \le y \le R/h_n$, Cauchy–Schwarz gives

$$|c_n(y) - c_n(L)| \le CN\left(L^{-13/2} + h_n^4 L^{-5/2}\right).$$

This is a bound on the total variation in the outer tail, not merely on each fixed compact interval.

At the original endpoint, $C_{n,2}(R) = \zeta C_{n,1}(R)$. Since $z_n(R) = 0$, the difference between corrected and uncorrected coefficients there is $-\nu \mathsf{M}(R) z_n'(R)/(i\tau_n)$. The fixed-collar trace bound in Proposition 9.54 gives

$$h_n^{-1/2} |\tilde{C}_n(R) - C_n(R)| \le C h_n^{-1/2} |\tau_n|^{-1/4} N = C h_n^{5/4} N.$$

Combining the last two estimates proves the displayed reflection defect bound.

Pass to a locally convergent subsequence with constant σ. If $\sigma = 1$, Proposition 9.55 gives $c = (Ae^{-x^{-7}}, 0)$; the reflection estimate then gives $|A| e^{-L^{-7}} \le CNL^{-13/2}$ for each $L \ge 1$. Letting $L \to \infty$ gives $A = 0$. The other sign gives the same conclusion with exchanged components. Since every subsequence has a further locally convergent subsequence with zero limit, the whole sequence converges locally uniformly to zero.

The variation bound implies

$$\limsup_n |c_n(R/h_n)| \le CNL^{-13/2}$$

for every fixed L, so the outer values themselves tend to zero. Integrating backwards from that endpoint yields, for $x \ge L \ge 1$,

$$|c_n(x)| \le |c_n(R/h_n)| + CN(x^{-13/2} + h_n^4 x^{-5/2}).$$

The resulting weighted tail bound is

$$\int_L^{R/h_n} x^{-2} |c_n|^2 dx \le C\left(L^{-1} |c_n(R/h_n)|^2 + N^2 L^{-14} + N^2 h_n^8 L^{-6} \right).$$

Use local uniform convergence on $[\varepsilon, L]$, then let $L \to \infty$. This proves the final assertion. ▫

Thus the pressure relation is no longer an open endpoint obligation for these limits. The remaining norm question cannot be answered by declaring one of the locally compact amplitudes nonzero: Proposition 9.56 proves they all vanish.

11. Physical norm outside the smaller axis region [EXACT]

The exact corrected coefficients also admit a differential reconstruction. Define

$$\mathsf{A}(r) = \begin{pmatrix} 0 & 2d/(\gamma r^3) \\ -d\kappa_E/(2\gamma r) & 2/r \end{pmatrix}, \qquad \mathsf{V}(r) = \begin{pmatrix} -\beta^2/r^4 & 8d/(\gamma r^4) \\ -3d\kappa_E/(\gamma r^2) & -\beta^2/r^4 + 8/r^2 \end{pmatrix}.$$

The algebraic identity of §8 and the definition of $\tilde{C}$ give, exactly on every collar,

$$\left[-\nu \mathrm{diag}(\Delta_4, \Delta_2) - \nu \mathsf{A}\partial_r - \nu \mathsf{V} + i\tau\gamma r^2 \right] z = f + i\tau\gamma r^2 \mathsf{W}\tilde{C}.$$

Indeed $\mathsf{A} = \gamma r^2 \mathsf{L}_0^{-1} \mathsf{D}$ and $\mathsf{V} = \gamma r^2 \mathsf{L}_0^{-1} (\mathsf{D}\mathsf{N}/r - \mathsf{D}' + \mathsf{B}\mathsf{D})$. This is an identity for the original solution and its actual coefficients; it introduces no independent boundary condition or glued inverse.

Write $a = b/\gamma$ (so $a^2 = \beta^2$), $\omega = \tau\gamma h^7 \ne 0$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P = \mathrm{diag}(0, 1), \quad D_0 = \mathrm{diag}(3/4, -29/4), \quad K_0 = \begin{pmatrix} 0 & -7/2 \\ 9/2 & 0 \end{pmatrix}.$$

Let $\mathsf{B}_h(x) = h\mathsf{H}(hx)^{1/2} \mathsf{W}(hx)$; this matrix is distinct from the primitive-system matrix B. The isometric rescaling in §9 gives the exact equation

$$-\nu h^3 v'' - \nu\left(h^2 a x^{-2} J + 2h^3 x^{-1} P \right) v' + \nu\left(h\beta^2 x^{-4} I + h^3 x^{-2} D_0 + h^2 a x^{-3} K_0 \right) v + i\omega x^2 v = i\omega x^2 \mathsf{B}_h c + h^5 w.$$

Here $|\omega| = \nu b^3/(7|\gamma|^3)$ is independent of h, and $v(R/h) = 0$. The norm matrix in §8, extended over the remaining compact interval, gives

$$|\mathsf{B}_h(x)q| \le Cx^{-1}|q|, \qquad 0 < x \le R/h.$$

Proposition 9.57 (physical mass escapes below the transport scale). Under the hypotheses of Proposition 9.55, for every fixed $\varepsilon > 0$,

$$\| \mathbf{1}_{r \ge \varepsilon\rho_{|\tau_n|}} z_n \|_{\mathcal{H}} \to 0.$$

In particular, any normalized quasimode sequence would concentrate at $r = o(\rho_{|\tau_n|})$.

Proof. Choose a real smooth χ equal to zero for $x \le \varepsilon/2$ and one for $x \ge \varepsilon$. Test the exact rescaled equation against $\chi^2 x^{-2}\bar{v}$ and integrate to R/h. There are no endpoint terms: χ vanishes at the inner end and $v(R/h) = 0$. Put

$$E_h = h^3 \int \chi^2 x^{-2} |v'|^2 dx, \qquad M_h = \int \chi^2 |v|^2 dx.$$

The two main terms in the pairing are $vE_h + i\omega M_h$. Every other term on the left, including the derivative of $\chi^2 x^{-2}$, has absolute value at most

$$\delta E_h + C_{\varepsilon,\delta} h \| v \|_2^2, \qquad 0 < h \le 1.$$

For example, the off-diagonal first derivative is bounded by

$$Ch^2 \int \chi^2 x^{-4} |v'||v| \, dx \le \delta E_h + C_\delta h \int \chi^2 x^{-6} |v|^2 dx.$$

The scalar potential contributes $O(h \int \chi^2 x^{-6}|v|^2)$; the other potential and derivative terms carry h^2 or h^3 and have the same stated bound since $x \ge \varepsilon/2$ on their support. The forcing pairing is bounded by

$$|\omega| \, \| \chi \mathsf{B}_h c \|_2 \, M_h^{1/2} + C_\varepsilon h^5 \, \| w \|_2 \, M_h^{1/2}.$$

Since $|vE_h + i\omega M_h| \ge c(E_h + M_h)$, choose δ small and apply Young's inequality to obtain

$$E_h + M_h \le C \int_{\varepsilon/2}^{R/h} x^{-2} |c|^2 dx + C_\varepsilon h \| v \|_2^2 + C_\varepsilon h^{10} \| w \|_2^2.$$

All constants are independent of the moving outer endpoint. Proposition 9.56 makes the first term tend to zero; the other terms tend to zero by boundedness. The isometry of §9 identifies $\int_\varepsilon^{R/h} |v|^2$ with the asserted physical norm squared. ▫

The cutoff here tests one exact differential identity. It does not localize the Volterra operator or patch representation-specific symmetrizers. It closes physical norm reconstruction outside every fixed inner fraction of the transport scale. Its constants depend on ε; letting $\varepsilon = \varepsilon_n \to 0$ without further estimates is not justified.

12. The inner gap is closed in the continuation [EXACT]

Propositions 9.56–9.57 leave only possible concentration at $r = o(|\tau_n|^{-1/7})$. Attachment D excludes it: Proposition 9.58 removes every fixed core $r \le B|\tau|^{-1/6}$, Lemma 9.59 estimates the exact six-component system on the intervening interval, and Theorem 9.60 controls both endpoint states and exhausts the physical norm. Thus bounded state and residual norms imply $\| z_n \|_{\mathcal{H}} \to 0$. Full frozen-window high-frequency compactness is proved; no exchange of the limits in Proposition 9.57 is used.

Where the present absorption ceases to be uniform [BAL / method limitation]. Its largest displayed potential relative to the imaginary mass has size $C/(|\tau| r^6)$, or Chx^{-6} in the rescaled equation. It becomes order one at $r \asymp |\tau|^{-1/6}$, which lies below $\rho_{|\tau|}$. This identifies a limit of the present perturbative estimate, not a new kernel or a new independent subsystem. The smaller-scale argument must retain the diffusion and pair wherever this ratio is not small.

The local cutoff calculation in §3 cannot supply this reduction, and the transport estimate alone does not establish compactness. Theorem 9.60 supplies the additional core and intermediate estimates with actual endpoint and norm control. No global weak inboard solution, frozen-window surjectivity, all-frequency non-resonance or full height result is claimed here.