# The Exterior Exponent of a Collapsing Vortex

### Attachment A: a relational decomposition of the mean–interaction architecture in three-dimensional Navier–Stokes

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*Attachment A to Structural Classification of the Exterior-Exponent Collapse Mechanism (Paper I), 2026.*

> **How to read this attachment.** This is the complete technical manuscript behind Paper I, which cites its results by the theorem, lemma and section numbers used here. The manuscript presents the analysis in the order it was carried out. Intermediate statements that later sections corrected are kept, and marked in place as withdrawn, superseded, failed or redundant, so that each correction can be checked. The claim ledger accompanying Paper I gives the current status of every claim and takes precedence where the two differ. Status labels are defined in Paper I, §1 and in the claim ledger. Bracketed citations refer to the reference list of Paper I. Labels such as cv25 identify symbolic and numerical runs whose scripts are in the calculations repository [13].

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## Abstract

We study a class of candidate finite-time singularities of the three-dimensional incompressible Navier–Stokes equations in which an axisymmetric collapsing mean is coupled to a non-axisymmetric interaction field of zero angular mean, whose quadratic momentum fluxes act on the mean as a Reynolds stress. For this architecture we show that the swirl amplitude, the axial contraction rate and the finite-energy condition are all determined by a **single exterior datum**: the local power-law exponent $\alpha$ of the ambient circulation at the edge of the collapsing core.

Three exact identities govern the reduction. The core width obeys $d\rho^2/dt = -2\sigma\rho^2 + 4\nu$ exactly, so radial contraction is $K := \sigma\rho^2/\nu > 2$ and there is no universal parabolic floor on $\rho$. The full angular-momentum identity carries **two** obstructions to the swirl maximum principle, one inviscid and one viscous, so that principle is supplied by exact axisymmetry of the *velocity* and not of the pressure. And the mean swirl equation carries the Reynolds stress as an exact flux divergence, so the interaction field redistributes angular momentum without creating any.

The reduction turns on a decomposition rather than a calculation. The decomposition is chosen with a small set of explicit rules about which objects carry their own state and where a coupling term belongs [12]; five of them do identifiable work, and each is named where it acts (§1.3). The rules choose what to inspect, never what is true: every statement below is settled by the Navier–Stokes equations and is stated with its premises. Applied to an earlier analysis of the same architecture, they overturned its decomposition (§§1.4, 5.3).

The substance of the decomposition is this. Within the class of source-free leading cores (A1b), a load-bearing test — a term belongs to the governing balance of a subsystem only if it is nonzero there — places the Reynolds stress in the **matching problem between the core and the exterior**, not in the core; the test classifies, and (A1b) is the premise. On the core the stress then vanishes, the mean swirl equation reduces to advection and diffusion with no source, and the maximum principle then gives

$$\Lambda \;\le\; \max_{\partial_{\mathrm{par}} C} \bar{\Gamma} \;\asymp\; \Gamma_{\mathrm{ext}}(\rho),$$

a theorem in place of a model: **the core cannot make swirl, it inherits it through its own boundary.** With $\alpha := -d\log\Gamma_{\text{ext}}/d\log r$ at the core edge, the exact width law then yields $G = \alpha(1 - 2/K)$ for the swirl growth rate in strain units, and the axial balance yields $\eta = -(1-\alpha)(1-2/K)$ for the axial contraction rate — **negative for every** $\alpha < 1$. The swirl exponent is $a = \alpha/2$; the scalar evolution equation for $K$ becomes an identity; and $K$ cancels from every admissibility condition, leaving

$$\text{within } K > 2: \qquad \Lambda \text{ grows} \Leftrightarrow \alpha > 0, \qquad E_{\text{core}} \to 0 \Leftrightarrow \alpha < \frac{1}{3}.$$

The cancellation is required, not convenient: $K$ depends on the radius at which one draws the core boundary, and a physical criterion may not depend on an arbitrary cut. What the window does **not** contain is realizability: whether the stress the matching problem demands lies in the cone the interaction field can supply.

That question is largely answered here. The cone is computed in closed form and is **circular, with its axis the radial shear**, so the shear sets both the demand and the cone; pointwise realizability then fails only on a measure-zero ray. Its nondegeneracy parameter turns out to be the exterior exponent itself — in the exterior $v_s - 2 = \alpha$ exactly — so the cone condition and the window condition are **one condition**, reached independently from a covariance constraint and from a kinematic argument. The cone does not degenerate as $\alpha \to \frac{1}{3}$; it degenerates at $\alpha \to 0$. Uniformity is structural, the matching interval being compact and the exterior margin constant. An attempted invariant-region proof then yields a **structure theorem** rather than a proof: the demanded stress obeys a *driven linear* system whose sources carry axial derivatives no radial state records, so **radial state** $\neq$ **profile state**, there is no invariant region, and the problem is a constrained design problem — infinitely many degrees of freedom, finitely many constraints. In that formulation the cone conditions define an open set, a moment discrepancy $\mu$ can be repaired without eating a margin $\delta_0$ in a region of width $W$ exactly when $\mu \ll \delta_0 W^2$, and the moment map — fibred over the axial coordinate — has a fibrewise Jacobian that is **triangular on the midplane** with cross term $O(\alpha\eta)$, hence a submersion. What remains is the existence of a **single near-solution** inside the cone-admissible set; the correction to an exact solution is then automatic.

That remaining step is then reduced again, and settled in its linear form. Linearising the residual map about a fixed profile — the linearisation is computed in closed form and validated against the nonlinear map to $10^{-8}$ — turns cone membership into a **linear programme**, because $T \in \mathcal{C}(s)$ is equivalent to two linear inequalities once $s$ is fixed. That programme is feasible: in every one of 64 admissible backgrounds spanning $\alpha \in (0, \frac{1}{3})$ and four layer positions there is a moment-preserving perturbation, supported in the layer, whose linear residual lies strictly inside the cone at **every** point of the layer with a uniform angular margin. Consequently the cone is **numerically eliminated as an obstruction across the tested family, and analytically reduced to a linear cone-intersection theorem**: modulo that theorem, Lemma $H''$ follows from the existence of one admissible profile whose residual vanishes on a layer carrying $\alpha_{\text{loc}} > 0$. The remaining object is then constructed. Writing out what vanishing stress *means* gives a first-order equation for the circulation slope, and that equation carries an obstruction of its own: at $l = 0$ it forces $l' = h/2 > 0$ whatever else the profile does, so **a stress-free region cannot generate a positive exterior exponent — it can only inherit one across its boundary**, which is the supply theorem again, one level down. Requiring the layer to *hold* a constant exponent then forces the radially integrated axial flux to equal $2 + \alpha$ — the same constant obtained independently from the exterior axial balance. At the resulting base the perturbation is closed-form and turns out to be **finite rather than tangent**: because the averaging operator is linear and the source is linear in it, the residual is *exactly* $\varepsilon(\chi, 0)$ for every $\varepsilon$, with both moments preserved identically and the cone itself unmoved, so it lies on the cone axis with no smallness condition. The one scalar condition left on the base — that the inboard profile enter the layer with vanishing stress — is then solved, not shot: it is a strictly monotone scalar functional of the inboard circulation undershoot,

with a **unique** zero. **Lemma $H''$ therefore holds on the active layer of the midplane fibre $\eta = 0$, analytically**, with strict cone membership on the open active layer and a demanded stress that must vanish at the layer edges — itself a consequence of the moment condition rather than a choice of test function. The one caveat is transcription: every step is algebra performed on the transcribed source equations. Testing the continuation off the midplane then corrected the paper twice more. At the corners, an earlier version's conclusion that the layer endpoints must move with height does not survive: the first transverse jet of the demand is reached by the axial curvature of the pressure, which the first-order axial moment condition forces to be nonzero in any case; every finite Taylor truncation of the natural continuation returns a cusp, because each order in height costs three radial derivatives of the demand, so no truncation decides the geometry; and an edge design in which the stress near each corner is carried by the circulation profile instead of by axial flux has **fixed edges** exactly, on full rectangles in the radius–height plane, conditional on four scalar matching identities in height. More seriously, the same inspection showed that the midplane closure is **local**: with zero axial velocity on the midplane, the axis-regular preparation region inboard of the layer necessarily carries demanded stress where no cone exists. Reflection symmetry is therefore excluded for the global problem and a midplane axial velocity is necessary — the design choice the source construction makes, for its own stated reason. Restoring that velocity does not by itself close the midplane. With the circulation profile kept even to first order in height, the preparation lifts exactly from the source equations and is strictly cone-admissible, with the demand on the cone axis, from the axis collar through the zero of the circulation slope; but it cannot stay on-axis all the way to the layer, because any preparation whose axial stress is slaved to its axial shear forces the midplane axial velocity to vanish identically. That step turns out to be unnecessary. Allowing the circulation profile an odd part at first order in height makes the whole region inboard of the layer exactly stress-free, by a linear lift for the axial flux whose only obstruction is a resonance of an indefinite Sturm–Liouville problem. An energy identity with a Hardy bound excludes that resonance whenever the midplane axial velocity exceeds an explicit threshold on the inner region, and the same bound makes the problem coercive. The midplane demanded stress is therefore zero or strictly cone-admissible at every radius — analytically, up to transcription. Below the threshold, resonances do occur, at a discrete sequence of amplitudes. Each correction is an artifact of a restriction adopted for tractability or of a closure checked locally rather than globally. One obstruction has now been found, and it is to the reflection-symmetric class, not to the architecture.

The reduction is compared against the construction of OpenAI's *Finite Time Blowup for Navier–Stokes* [2], which realizes this architecture. Every relation above is satisfied there, several identically, with $\alpha = 2h$ and $h < 1/100$. **This is a consistency check and not a prediction.** The reduction of §§6–8 was derived *after* that construction had been examined, so the agreement cannot be read as a successful forecast; what it establishes is that the reduction is non-vacuous, that its exact relations hold for an object built on different premises, and that the open lemma has at least one solution. A separate and weaker independence claim, concerning the architecture and the $(K, G, \eta)$ ledger rather than the exterior exponent, is stated with its date in §1.4. The construction has not yet been independently reviewed and confers no authority here. **No priority is claimed for anything.**

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# 1. Introduction

## 1.1 The architecture under study

Let $u$ solve the incompressible Navier–Stokes equations on $\mathbb{R}^3$,

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \qquad \nabla \cdot u = 0,$$

and suppose a singularity is approached at the spatial origin as $t \uparrow T$. Write $s = T - t$ and use cylindrical coordinates $(r, \theta, z)$.

Decompose by the azimuthal average $\bar{\cdot}$:

$$u = \bar{u} + u', \qquad \bar{u'} = 0, \qquad \tau := \overline{u' \otimes u'}.$$

The class of candidates studied here is characterised by three features, stated as standing hypotheses rather than assumed silently:

> **(A1) Mean–interaction architecture.** $\bar{u}$ is axisymmetric and carries a collapsing vortex; $u'$ is a non-axisymmetric field of zero angular mean whose quadratic products $\tau$ have nonzero angular mean.
>
> **(A1b) Source-free leading core [CLASS].** The interaction stress vanishes at leading order in the interior of the self-similar core, $\mathcal{T}[\tau] = 0$ on $\mathcal{E}_1$; any nonzero stress belongs to the compatibility region joining the core to the exterior. This is a restriction on the class and not a consequence of the equations — (S′) holds for any $\tau$. Where the compatibility region begins is not fixed by (A1b); §9.16.8(g) and §9.16.9 bear on it.
>
> **(A2) Shrinking core.** There is a core region $C_s$ of radial width $\rho(s) \to 0$ and axial length $\ell(s) \to 0$, with $\ell \gg \rho$, on which the mean swirl $\bar{\Gamma} := r\bar{u}^{\theta}$ attains its maximum $\Lambda(s) := \| \bar{\Gamma} \|_{L^\infty(C_s)}$.
>
> **(A3) Ambient power law.** Outside the core the circulation has a local power-law behaviour, $\Gamma_{\text{ext}}(r) \sim C_\infty r^{-\alpha}$ for $r$ slightly larger than $\rho$, with $\alpha$ fixed.
>
> **(A4) Saturated supply.** The bound of §6.1 is attained: $\Lambda(s) \asymp \Gamma_{\text{ext}}(\rho(s))$, not merely $\leq$. Equivalently, the mean swirl at the core edge is not asymptotically smaller than the ambient profile there.

Nothing is assumed about the sign, phase or amplitude of $\tau$. Its support is assumed only to the extent of (A1b); everything else about it is an output.

## 1.2 What is claimed

| | statement | status |
|---|---|---|
| **Not claimed** | that the equations develop a singularity, or that they do not | — |
| **Not claimed** | that this architecture is realized, or that it exhausts the possibilities | — |
| **[EXACT]** | the width law (§2), the two-torque angular-momentum identity (§3), and (S′) with $\mathcal{T}[\tau]$ a flux divergence (§4) — consequences of the equations and incompressibility alone | §2–§4 |
| **[CLASS]** | (A1b) source-free leading core — a class premise; R9 classifies the stress given it and does not prove it | §1.1, §5.2 |
| **[EXACT \| A1b]** | the supply theorem $\Lambda \leq \max_{\partial_{\text{par}} C} \bar{\Gamma}$, given (A1b) and axisymmetry of the core | §6.1 |
| **[COND \| GROWING-SUPPLY]** | that non-axisymmetry is necessary and cannot be confined to the core — needs $\alpha > 0$ **and** (A4) | §6.2(ii) |

| | statement | status |
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| **[EXACT \| SUPPLY]** | $G = \alpha(1 - 2/K)$ — "SUPPLY" is **(A4)**, since the derivation differentiates (6.1) as an equality | §7.1 |
| **[EXACT \| AXIAL-BALANCE]** | $\eta = -(1-\alpha)(1-2/K) < 0$, $a = \alpha/2$, and the identity $K(1+\eta-G) = 2$, given additionally (A5) | §7.2–§7.3 |
| **[COND]** | within $K > 2$: $\Lambda$ grows iff $\alpha > 0$; the core energy vanishes iff $\alpha < \frac{1}{3}$ | §8 |
| **[EXACT]** | the cone in closed form; pointwise realizability (Theorems 9.1–9.2) | §9.2–9.3 |
| **[EXACT]** | $a = 2 + \alpha_{\mathrm{loc}}$; in the exterior $v_s - 2 = \alpha$ — **the cone condition is the window condition** (Theorem 9.3) | §9.4 |
| **[EXACT]** | uniform margin: compactness on $I$, constant margin on the tail | §9.5 |
| **[EXACT]** | $D_X w_1 + l w_1 = \Sigma_q$, $D_X(E w_2) = \Sigma_n$: the stress system is linear and **driven** | §9.6 |
| **[EXACT]** | **radial state $\neq$ profile state**: the margins depend on axial derivatives no radial state carries, so no invariant region exists | §9.6 |
| **[EXACT]** | $\mathcal{U}$ is open in $C^1$; repair versus margin, $\mu \ll \delta_0 W^2$ (Theorem 9.5) | §9.8 |
| **[COND]** | $D_\Phi \mathcal{M}$ **is** a submersion: triangular at $\eta = 0$, cross term $O(\alpha\eta)$ (Theorem 9.6) | §9.9 |
| **[NUM]** | $\mu$ drives to machine zero; $M_0, M_2 > 0$ once the layer sits beyond the maximum | §9.11 |
| **[EXACT]** | the $\eta$-independent midplane family is **obstructed**: $w \to -s$, $M_1 < 0$ (Theorem 9.7) | §9.11 |
| **[EXACT]** | the axial flux must satisfy $c_U'(0) = 2 + \alpha$ (Corollary 9.8) | §9.11 |
| **[EXACT]** | $c_U(\eta) = (2+\alpha)\eta$ in closed form; $K = La$ identically | §9.13 |
| **[EXACT]** | the odd-flux profile degenerates at $\eta = 0$ (§9.13(i)); the necessity of $v := U(\cdot, 0) \neq 0$ once drawn from it is **withdrawn as a proof** — $v = 0$ is allowed on the active layer, and $v \not\equiv 0$ inboard is required globally by **Theorem 9.18**, which is the proof | §9.13, §9.16.8(g) |

| | statement | status |
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| **[EXACT]** | $w$ is a departure from zero-stress; the remaining step is a **linear** cone-feasibility problem | §9.13 |
| **[EXACT]** | the linearised residual map at $\eta = 0$ in closed form, validated to $10^{-8}$ against the nonlinear map | §9.15 |
| **[NUM]** | **Theorem 9.9**: a moment-preserving tangent perturbation supported in the layer, with $\delta w$ strictly inside $\mathcal{C}(s_*)$ at every point of $I$ (64/64 backgrounds) | §9.15 |
| **[COND]** | **Corollary 9.11**: given tangent feasibility at a zero-stress base, Lemma $H''$ follows | §9.15 |
| **[NUM]** | $\delta U_\eta$ must move, and independently of $\delta v$; the **cone**, not the moments, forbids tying them | §9.15 |
| **[NUM]** | **for the tested non-zero-stress family only**: the residual cannot be placed on the cone axis (feasible at $0.20\tan\theta_c$, infeasible at 0.15); at the zero-stress layer base Theorem 9.13 puts it exactly on the axis | §9.15, §9.16.7(iii) |
| **[EXACT]** | **Lemma 9.12**: a globally zero-stress profile has $\alpha_{\mathrm{loc}} < 0$ everywhere; the layer must *inherit* the exponent | §9.16 |
| **[EXACT]** | the stationary layer forces $\int_0^X U_\eta = 2 + \alpha$ — the same constant as Corollary 9.8, reached independently | §9.16 |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.13**: a **finite** – not tangent – cone-axis perturbation; $w = \varepsilon(\chi, 0)$ and both moments preserved for *every* $\varepsilon$ | §9.16.3 |
| **[EXACT \| TRANSCRIPTION]** | the demanded stress **must** vanish at both layer edges: $c_Q = 0$ forces it | §9.16.3 |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.14**: $F(\lambda) = w_1(X_a)$ is strictly monotone with a **unique** zero — the preparation parameter $\lambda_*$ is unique **within the chosen one-parameter family**, not the base across all admissible $(l_0, \beta, U_\eta)$; the numerical shooting is withdrawn | §9.16.4 |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.15**: $M_0, M_1, M_2 > 0$ on the open active layer of the midplane — a closure of the **layer**, not of the fibre (Theorem 9.18) | §9.16.5 |

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| **[CLASS]** | reflection symmetry is a class, not a consequence of $U(X,0)=0$; on the fixed $v=0$ layer/corner midplane, in either parity class, $w_2(\cdot,0)\equiv 0$ forces $\partial_\eta E\|_0 = 0$ — once $v\neq 0$, $E_1$ is an independent state (Prop. 9.19) | §9.16.8(a) |
| **[EXACT \| TRANSCRIPTION]** | **Corollary 9.16**: the first-order axial moment forces $\int E_0\,\partial_\eta^2 E\|_0 = 2\int (U^{(1)})^2 - 2\alpha\int \Pi_0 > 0$ — the pressure must bend in height | §9.16.8(c) |
| **[EXACT \| TRANSCRIPTION] + [NUM]** | the Theorem 9.13 continuation gives cusps $\|\eta\| < \kappa_b\|\xi\|^3$ and $\|\eta\| < \kappa_a\chi$; the earlier wedge is **withdrawn**; no finite truncation decides the geometry (three radial derivatives per order in $\eta$) | §9.16.8(d) |
| **[EXACT \| TRANSCRIPTION, given (G)] + [NUM]** | **Theorem 9.17**: circulation-driven edge zones give **fixed edges** on full rectangles in $(X,\eta)$ — moving endpoints are **not** forced | §9.16.8(e) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.18**: with $U(\cdot,0)\equiv 0$ the preparation region carries unrealizable demand; reflection symmetry is **excluded** for the global problem, and a midplane axial velocity is necessary inboard | §9.16.8(g) |
| **[EXACT \| TRANSCRIPTION]** | **Proposition 9.19**: with $v\neq 0$ the midplane stress depends exactly on $(E_0, v, U^{(1)}, E_1)$; $\Pi_0$, $U^{(2)}$, $E_2$ do not enter | §9.16.9(a) |
| **[EXACT \| TRANSCRIPTION]** | **Lemma 9.20**: a stress-free crossing of $l=0$ has $l' = \frac{1}{2}(h + vE_1/E_0)$ — Lemma 9.12 holds only where $vE_1 = 0$ | §9.16.9(b) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.21**: division-free source lift in the class $E_1 = 0$; the flux constant $\Psi(X_a) = 2+\alpha$ is automatic at the join | §9.16.9(c) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.22**: slaving obstruction — $w_2 = -\tilde{q}\,b_s$ on the whole preparation forces $v\equiv 0$; an exactly on-axis global preparation is impossible when $E_1 = 0$ | §9.16.9(d) |
| **[EXACT \| TRANSCRIPTION] + [NUM]** | **Lemma 9.23** (onset) and **Theorem 9.24** (shear bridge): exact on-axis admissible preparation from onset through the crossing | §9.16.9(d)–(e) |

| | statement | status |
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| **[FAIL]** | three families of return windows (cutoffs of the bridge profile; $q$ up to 8; steering of the axial-moment zero) rotate the demand out of the cone — evidence about those families, not an obstruction | §9.16.9(f) |
| **[FAIL]** | an intermediate corner computation omitted $-d\,\Pi_\eta$ from $S_n$ and concluded that the corners were unrepairable; withdrawn | §9.16.8(b) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.25** (stress-free lift): with $E_1 \neq 0$, a stress-free global midplane preparation is equivalent to one linear degenerate boundary-value problem for the axial flux $\Psi$; $E_1$ and $\Pi_1$ follow algebraically, and the joins, the flux constant and both midplane moments are automatic | §9.16.9(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.26** (non-resonance and existence): if $\min_{[0,\,X_0]} v^2 > Q := \int_0^{X_0} E_0^2\, l\, x^{-1} dx$, the §9.16.9(h) problem has exactly one solution, axis-regular and flat-matched at $X_s$ — **the global midplane closes**: demand zero or strictly cone-admissible at every radius | §9.16.9(h) |
| **[EXACT \| TRANSCRIPTION] + [NUM]** | resonance below the threshold is real: it occurs exactly when the Wronskian $\mathcal{W}[l, v]$ of the axis-regular and flat branches vanishes, i.e. at eigen-amplitudes of an indefinite Sturm–Liouville problem, each simple and transversal; for the constructed design $\lambda_* = 1.516$, with resonances at $\lambda = 1.2348,\ 0.5492,\ 0.3573, \ldots \to 0$ | §9.16.9(h) |
| **[REDUNDANT]** | the Return Lemma — not proved, not ruled out, and no longer needed | §9.16.9(g) |
| **[EXACT \| TRANSCRIPTION]** | the general endpoint/moment system (G) has **rank four**: outboard of the layer the exterior is stress-free for every $\eta$, so $c_Q = LHw_1(X_b, \eta)$ and $c_N = LEw_2(X_b, \eta)$ **[REDUNDANT pair]**. Within the stress-free inboard class it is **two scalar outer-edge conditions plus one functional stress-free PDE constraint**, since $w(X_a, \eta) = 0$ is then automatic | §9.16.8(h) |

| | statement | status |
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| **[EXACT \| TRANSCRIPTION] + [NUM]** | **Theorem 9.27** (height marching): the stress-free equations are parabolic in $\eta$, $\partial_\eta(\log E, U) = (2LX/H_c)\,\partial_X^2(\log E, U) + \ldots$, with $H_c = D\eta + dU$; at every order the new jets solve the operator of Theorem 9.25, forced by two radial derivatives of the previous jets | §9.16.8(h) |
| **[FORMAL]** | the inboard height hierarchy is solvable to every order under (NR), with no new resonance — but it is of heat type and diverges for generic data | §9.16.8(h) |
| **[FAIL]** | continuation of the closed midplane as an **initial-value problem** in $\eta$: backward-parabolic where $\eta\, v < 0$, and the stagnation surface $H_c = 0$ contains the midplane beyond $X_s$ and meets every neighbourhood of it as $X \to X_s$ | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Proposition 9.28** (stagnation geometry): on $H_c = 0$ the stress-free equations lose their local $\eta$-derivatives but keep radial integrals along the *horizontal* slice, so a non-horizontal $\Sigma_*$ carries no intrinsic trace problem; $\Sigma_*$ is a two-sided forward source iff $\partial_\eta H_c > 0$, and a sink otherwise; $H_c(X, \pm 1) = \pm D$, so both ends of the strip are outflow | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.29** (no horizontal stagnation slice): if $H_c \equiv 0$ on a slice $\eta = \eta_0$ over the radii where $l \geq 0$, the demand on that slice is not zero or admissible at every radius — Theorem 9.18 at height $\eta_0$, with $l' = h/(2d)$ at a stress-free zero of $l$ | §9.16.8(h) |
| **[FAIL]** | the stagnation surface as a two-sided **design manifold**: horizontal pieces are excluded inboard (Theorem 9.29), non-horizontal pieces carry matching rather than data, and on the plateau the closed midplane lies below a local sink ($v_0 = 3$, **[NUM]**) and above a source at $O(1)$ depth | §9.16.8(h) |

| | statement | status |
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| **[EXACT \| TRANSCRIPTION]** | **Proposition 9.30** (intrinsic clock): with $\tau_\eta = 1/H_c$ the principal operator is $2LXV^2 - \partial_\tau$, where $V = \partial_X\|_\eta = \partial_X\|_\tau + \tau_X\partial_\tau$. Each sign-domain of $H_c$ is forward-parabolic along its own drift, so "backward in $\eta$" is an orientation artifact; $\|\tau\| \to \infty$ at every zero of $H_c$; but $\tau$ is not a Cauchy clock, since $V\tau = \tau_X \to \infty$ at a non-horizontal $\Sigma_*$ | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.31** (when the sign-domains separate): where $\Sigma_*$ is horizontal the diffusion leaves $\eta =$ const do not cross it, and the two sides are separate boundary-value problems at infinite intrinsic depth — joined by equality of traces at a source, overdeterminedly at a sink. Where $\Sigma_*$ is transverse to the leaves, they cross it at finite radial distance through a non-degenerate radial operator, and the two sides are **one boundary-value problem** | §9.16.8(h) |
| **[FAIL]** | the **intrinsic-clock decomposition** inboard: Theorem 9.29 excludes horizontal $\Sigma_*$ there, and for the closed design every leaf with $\eta < 0$ crosses $\Sigma_*$ between the last pole of $\eta_*$ and $X_s$, coupling the lower outer lobe to the inboard region (**[NUM]** sign pattern) | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | the global forward–backward boundary-value problem is the **invariantly correct** inboard problem — not merely the current endpoint of the manuscript. Its genuine singular set is the sinks of the drift, the tangencies of $\Sigma_*$ with the leaves, and the corner $(X_s, 0)$ | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Proposition 9.32** (stagnation Gauss map): on $\Sigma_*$, source $\Leftrightarrow D(1+\eta^2)/d + dU_\eta > 0$ and leaf tangency $\Leftrightarrow U_X = 0$. Sinks and tangencies are the two walls of **one** condition on the direction of $\nabla H_c$ (R15: merged), and the index count forced by $H_c(X, \pm 1) = \pm D$ forces neither | §9.16.8(h) |

| | statement | status |
|---|---|---|
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.33** (forced singular points): with Cartesian axis regularity, $\Sigma_*$ meets the axis in a **fold tangency** with the leaves. It meets the horizontal exterior piece at $(X_s, 0)$ **infinitely flatly**, and straightening $\Sigma_*$ there gives the horizontal-source model $D\,t\,u_t = c\,u_{\xi\xi}$ plus flat terms. The singular set cannot shrink below these two points | §9.16.8(h) |
| **[NUM]** | near-midplane sinks ($D + U_1 < 0$ at $\lvert\eta_*\rvert < 0.05$) occur in all six tested closed-midplane lifts. They move with the return of $v$ but are not removed by lengthening the return or lowering its amplitude. Mechanism: $U_1 \simeq -(2X^3/hE_0^2)(vv''')'$ in the degenerate zone | §9.16.8(h) |
| **[OPEN]** | whether the sinks can be designed away. The geometry of $\Sigma_*$ is **state** of the determined boundary-value problem, controlled only through the handover trace, so the geometry-control map is the linearised solution operator | §9.16.8(h) |
| **[EXACT]** | Test A: $\int_{-1}^{1} \partial_\eta H_c\, d\eta = 2D$ and $\int_0^{X_a} \log(X_a/X)\, dX = X_a$ give $\sup_\eta B(\eta) \geq DX_a$, so the scalar condition $(\mathrm{NR}_\eta)$ forces $X_a < 4(1-\alpha)/D = 8$ **for every** $\alpha$. It does not exclude (NR) | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.34** (full energy identity): in the linearised stress-free system the transport divergence is $\partial_X(XW) + \partial_\eta H_c = (1+D)(1-2\eta U)$, **free of** $U_\eta$. The budget carries $+dU_\eta$, so **sinks consume it** — the scalar model's sign was an artefact of dropping the radial drift. The three $\eta$-derivative couplings $dlA_X(u_\eta)$, $dXU_XA_X(u_\eta)$, $dp_\eta$ are symmetrizable by bounded positive local diagonal multipliers only if $\mu l = \nu E^2$ (impossible across $l = 0$) and $U_X \equiv 0$; otherwise $\cos k\eta$ tests shift the form by $O(k)$. **No bounded positive local diagonal multiplier gives a coercive form, not even on smooth test functions**. Matrix, triangular and nonlocal symmetrizers are not covered | §9.16.8(h) |

| | statement | status |
|---|---|---|
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.35** (height mass operator): its Schur complement on every leaf is $(H_c^2\Phi')' + d^2E^2l\,\Phi/X^2$, the operator of Theorem 9.26 with $v \to H_c$ and $E_0^2 \to d^2E^2$. At the midplane $d = 1$ and $H_c = v$, so it coincides with that operator (trivially $d^2$ times it), with the same kernel, index and non-resonance condition; (NR) is its midplane slice. At a **simple** zero of $H_c$ the negative index is finite iff $\mathrm{Ri}_* = E^2l/(X^2U_X^2) \le \frac{1}{4}$ — the critical value included, by Hille–Kneser. At a tangency of order $m \ge 2$, where that formula does not apply, it is infinite iff $l > 0$ (Sturm–Liouville comparison). On the leaf through the forced axis fold it is **infinite** | §9.16.8(h) |
| **[NUM]** | all six tested closed-midplane lifts ( $R_{\mathrm{mid}} \in [1.5, 8]$, $X_s \in [3, 14]$) fail the scalar budget ($B(0) = 9.6$–$58.8$ vs 3.2) and every unit-multiplier band test. The scalar budget comes entirely from the return lobe of $U_1 > 0$, the system budget mostly from the axis and plateau. Amplitude barely moves the scalar budget | §9.16.8(h) |
| **[EXACT]** | **Theorem 9.36** (scalar axis fold): in $r = 2\sqrt{X}$, $(Xu_X)_X = \Delta_2 u$ is the Cartesian Laplacian; $H_c = \kappa t + \gamma r^2 + \dots$ with $\kappa$ the source function of Proposition 9.32 and $\gamma = dU_X/4$; at least one axis fold is a source. The model $(\kappa t + \gamma r^2)q_t = c\Delta q$ is well posed on small cylinders, needs **no data on** $t = \mathrm{const}$ **at a source fold**, and is hypoelliptic iff $\gamma \neq 0$ (Hörmander, step three). **Test A fails** | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.37** (coupled axis fold): at the axis, $A_X$ is the disc average, of degree 0 and not smoothing. The swirl–pressure pair is **supercritical** at the scalar-fold weight ( $t \sim r^4$) and is not a relatively bounded perturbation. At its own weight $t \sim r^3$ the fold coefficient is lower order, and the principal pair-plus-diffusion model reduces **exactly** to $u_{tt} = \frac{4L^2}{d^2\kappa_E}(-\Delta_2)^3 u$, which is quasi-elliptic of type (2, 6) and Cartesian-regular. It is well posed as a boundary-value problem in $t$ and ill posed only as a march. The infinite index of Theorem 9.35(c) is **not an instability** | §9.16.8(h) |

| | statement | status |
|---|---|---|
| **[FAIL]** | the class-level claim that radial diffusion outweighs the fold-leaf negative modes ( $\delta^{-2}$ against $\delta^{-1}$): the pair sets the principal model instead (Theorem 9.37(a)) | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.38** (modal structure): $\mathfrak{A}_\eta = H_c I + \partial_X^{-1}\tilde{K}$ modulo rank one and order $-2$. The mode masses are $m_\pm = \varepsilon(\zeta \pm \sqrt{1-\vartheta^2} + i\vartheta)$, with $\varepsilon = dE\sqrt{l}/(kX)$, $\zeta = H_c/\varepsilon$ and $\vartheta = (4\mathrm{Ri})^{-1/2}$. **The eigenvectors do not depend on $H_c$, $\zeta$ or $k$**; the balanced condition number is $\sqrt{(1+\vartheta)/(1-\vartheta)}$. The only coalescence is at $\mathrm{Ri} = \frac{1}{4}$ (a Jordan block), at $r \simeq 4\sqrt{\kappa_E}/\lvert U_X\rvert$, outside the fold neighbourhood. The transition $\lvert\zeta\rvert \sim 1$ carries no eigenvector transition | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.39** (mode zeros): at coarse scale both mode zeros are sources at a source fold, one per mode. At fine scale the shear imaginary part and the tilt are locked in ratio $\frac{1}{2}$, and each zero is the scale-invariant normal form $\mathcal{N}_{\mathrm{Ri}}$: $(x + c_*\partial_x^{-1})a_\tau = \nu a_{xx}$ with $c_* = -\frac{1}{2} \pm i\sqrt{\mathrm{Ri} - \frac{1}{4}}$ — **the Frobenius exponent of Theorem 9.35(b)**. Its exact solutions are $\int s^{c_*} e^{xs - \nu s^3/3\sigma} ds$, with connection constants $\Gamma(1 + c_*)$ bounded uniformly. Scale invariance makes every estimate uniform in $k$ | §9.16.8(h) |
| **[EXACT] + [NUM]** | core traces and radial conditions: the $(2, 6)$ core needs one $t$-trace per end, and each end receives exactly one incoming mode from a data-free source. These are **transmission traces**, in $H_x^{5/2}$. The inherited radial triple is $\{u,\ \Delta_2 u,\ \partial_r \Delta_2 u\}$ at $r = R$ (the $u$-trace, the pressure trace — nonlocal globally — and the $e$-trace). It is complementing: by factorisation at $\tau = 0$, and with minimal Lopatinskii singular value 0.213 on $\tau^2 + \lvert\xi'\rvert^6 = 1$ | §9.16.8(h) |
| **[FAIL]** | the **[CLASS]** statement of the former Theorem 9.37(e), read locally: the mode zeros are sources only at coarse scale; at fine scale they are $\mathcal{N}_{\mathrm{Ri}}$ | §9.16.8(h) |

| | statement | status |
|---|---|---|
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.40** (T1 closed): $(x + c\,\partial_x^{-1})\lvert D\rvert^c = \lvert D\rvert^c x$ exactly, so $\mathcal{N}_{\mathrm{Ri}}$ **is the scalar Kolmogorov equation** $xb_\tau = \nu b_{xx}$ under $a = \lvert D\rvert^c b$. The Richardson number is a gauge at fine scale. The nonlocal multiplier $\lvert D\rvert\bar{a}$ gives an energy identity with constant 1, uniform in $k$ and in $\mathrm{Ri} \geq \frac{1}{4}$. The half-range problem is uniquely solvable with **no data at the zero**, and there is no derivative loss ($\lVert a\rVert_{\dot{H}^{s+1/2}} = \lVert b\rVert_{\dot{H}^s}$) | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.41** (Hermitian fold; T2 and T3 closed for the symmetric fold): at the axis $l > 0$, so Theorem 9.34's condition $\mu l = \nu E^2$ **is** admissible with weights degenerating like $E^2$ — in $r$, the natural measures $\frac{\kappa_E}{4} r^3 dr$ for $e$ and $r\,dr$ for $u$. $H_c$ plus the swirl–pressure pair is then **self-adjoint**, with pair form bounded by $\frac{d\sqrt{\kappa_E} R}{2}\lVert Z\rVert_H^2$. One energy covers core, corner and outer fold at once. On windows with $2Lj_{0,1}^2/R^2 > \kappa/2$ and $\kappa T > \lvert\gamma\rvert R^2 + \frac{d\sqrt{\kappa_E} R}{2}$ the symmetric fold is well posed with **no height data and no resonance** | §9.16.8(h) |
| **[FAIL]** | the remark (Theorem 9.40, next theorem) that the corner is self-similar under $t \sim r$ and therefore a Mellin problem: at $t \sim r$ radial diffusion dominates by $\lambda^{-2}$. The corner's own scale is $t \sim r^3$ with $H_c$ frozen, and Theorem 9.41 needs no matching there | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Proposition 9.42** (the residue): the only principal term not made self-adjoint is the shear's skew Volterra mass $Su = -\frac{dU_X}{2}\int \mathrm{sgn}(X - x)u$, with $\lVert S\rVert_{\mathrm{HS}} \leq \lvert\gamma\rvert R^2/2$, acting on $u_t$. No bounded change of weights removes it: the pair-free Sylvester correction has kernel $\propto 1/\lvert X - x\rvert$, a logarithmic divergence. That is why Theorem 9.40 needed the order-one gauge $\lvert D\rvert$ | §9.16.8(h) |

| | statement | status |
|---|---|---|
| **[EXACT]** | **Lemma 9.43** (degenerate masses need exact symmetrizers): if any multiplier $G(t)$ , local or not, makes the energy form coercive on smooth test functions in a norm without $t$-derivatives, then the skew part of $G^{*}\mathcal{A}(t)$ vanishes **identically**. Height-oscillating tests otherwise shift the form by $O(k)$. No approximate symmetrizer works, however small or compact its error | §9.16.8(h) |
| **[FAIL]** | the glued symmetrizer of Proposition 9.42(c), whose cut-off errors are skew mass terms; and Theorem 9.40(d)'s "localisation costs a compact error", which holds only outside the mass | §9.16.8(h) |
| **[NUM]** | the exact-symmetrizer landscape. With shear the fold mass has exactly real spectrum, and the order-½ similarity growth matches the proven normal form. The optimal exact symmetrizer and the Laplacian-power family both fail coercivity or exactness **on the fold and on the proven control alike**. Inconclusive about (S); conclusive that exactness is a whole-line property destroyed by windows | §9.16.8(h) |
| **[EXACT] + [NUM]** | **Proposition 9.44** (invertibility in height frequency, no multiplier): $\|(\mathcal{L}+i\tau\mathcal{A}_0)z\|_H \geq \lambda_1\|z\|_H$ for all $\tau$ in the symmetric fold, and $\geq \frac{\lambda_1}{2}\|z\|_H$ for $\|\tau\| \leq \lambda_1/(2\|S\|)$ with shear. Numerically $\min_\tau \sigma_{\min}/\lambda_1 = 1.0000$ with shear on, at every resolution and for the control. **The window obstruction is a method obstruction, not a PDE obstruction** | §9.16.8(h) |
| **[FAIL / REDUNDANT]** | the glued-symmetrizer route (S). It is excluded by Lemma 9.43 and superseded by operator-level quasimode exclusion | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Lemma 9.45** (frequency confinement of quasimodes): $\mathrm{Re}\langle\mathcal{R}(\tau)z,z\rangle_H \geq \langle\mathcal{L}z,z\rangle_H - C_s\|\tau\|\,\|\mathcal{L}^{-s/2}z\|_H^2$ for every $s<\frac{1}{2}$, since the skew shear gains $2s$ derivatives. Normalised $\varepsilon$-quasimodes therefore carry $\mathcal{L}$-spectral mass at most $(\varepsilon + C_s\lambda_1^{-s}\|\tau\|)/\Lambda$ above $\Lambda$ | §9.16.8(h) |

| | statement | status |
|---|---|---|
| **[EXACT \| TRANSCRIPTION]** | **Proposition 9.46** (local model list; exhaustion conditional): four proposed local limit operators have trivial kernels on their specified domains — (i) a regular point, symbol $\nu\xi^2 + i\tau m$ with $m \neq 0$; (ii) a mode zero, $\mathcal{N}_{\mathrm{Ri}}$, which **after** blow-up the whole-line gauge of Theorem 9.40 turns into the complex Airy operator $-\nu\partial^2 + i\tau x$ , with no tempered kernel, uniformly as $\mathrm{Ri} \to \infty$; (iii) the axis core, symmetric at that scale and quasi-elliptic; (iv) the boundary $r = R$, where no mode zero reaches at high frequency and the frozen mass $\gamma R^2 \neq 0$. The $\tau^{1/4}$ band is the complex-oscillator scale of the double zero of $\gamma r^2$, and the pair destroys it | §9.16.8(h) |
| **[NUM]** | large-$\tau$ growth law (cv37): $\sigma_{\min}(\mathcal{L} + i\tau\mathcal{A}_0)$ rises from 26 at $\tau = 10^2$ to 4260 (shear off) and 3065 (shear on) at $\tau = 10^6$, identical at $N_r = 200$ and 400. The local exponent is 0.55–0.56 and still rising; the minimiser is interior ($\langle r \rangle$ from 0.66 to 0.26); the complex-Airy control gives exactly 2/3. The constructed mode-surface family costs $\tau^{8/13}$, an upper bound that the true minimiser beats, so the classification exponent is not sharp | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Theorem 9.47** (exact mode decoupling at the axis). $D_1 = \frac{1}{r}\partial_r$ intertwines, $\Delta_4 D_1 = D_1\Delta_2$ and $D_1^* D_1 = -\Delta_2$, and $\mathfrak{M} = 2D_1\Delta_2^{-1}$, $p = -\frac{\kappa_E}{2}\Delta_2^{-1}D_1^* e + \text{const}$. With the isometry $U = D_1(-\Delta_2)^{-1/2}$ and $a_\pm = (\tilde{u} \pm U^*\tilde{e})/\sqrt{2}$, the pair is **exactly** $\pm d\sqrt{\kappa_E}(-\Delta_2)^{-1/2}$ and the diffusion **exactly** $-2L\Delta_2$, on whole-space radial functions (windows add boundary defects). Only $\gamma r^2$ and the shear couple the modes | §9.16.8(h) |
| **[EXACT \| TRANSCRIPTION]** | **Lemma 9.48** (the mass has no kernel): $\mathcal{A}_0 z = 0$ reduces to $\Psi'' - \Psi'/r + \beta^2\Psi/r^4 = 0$, $\beta = d\sqrt{\kappa_E}/\gamma$, solved by $(r/\beta)Z_1(\beta/r)$. Every nonzero solution has $u \sim r^{-3/2} \notin L^2(r\,dr)$, so $\ker\mathcal{A}_0 = \{0\}$. The near-kernel oscillates with phase $\beta/r$, i.e. on the mode-zero curve $\gamma r^2\lvert\xi\rvert = d\sqrt{\kappa_E}$; that curve generates $0 \in \sigma_{\mathrm{ess}}(\mathcal{A}_0)$ | §9.16.8(h) |

| | statement | status |
|---|---|---|
| **[EXACT] + [COND]** | **Proposition 9.49** (compactness reduction). (a) **[EXACT]** Quasimodes bounded in $V$ are impossible: $V \Subset H$ and Lemma 9.48 force a zero limit. So every normalised quasimode escapes, with $1 \ll \langle \mathcal{L} z_n, z_n \rangle \lesssim |\tau_n|$ and $z_n \rightharpoonup 0$. (b) **[COND]** on standard semiclassical-measure theory: in the mode variables of Theorem 9.47 the mass is the Cartesian symbol $\gamma|x|^2 \pm d\sqrt{\kappa_E}|\xi|^{-1}$, so the anisotropic measure ($x \sim h^{1/2}$, $\xi \sim h^{-1}$) lives on that curve. At interior points, where the carrier $\gg$ blow-up frequency, Proposition 9.46(ii) excludes it; the isotropic core is covered by (iii), while this local argument does not itself establish two-scale exhaustion (now supplied by Theorem 9.60); the characteristic curve does not reach $r = R$ at high carrier frequency | §9.16.8(h) |
| **[EXACT]** | **Lemma 9.50** proves the whole-line normal-form lower bound through zero Fourier frequency. The unrestricted weighted Green operator fails through range compatibility. The proposed independent-model transfer (Q) is superseded, for full window compactness, by the direct proof of Theorem 9.60 | §9.16.8(h) |
| **[EXACT] + [BAL]** | **Propositions 9.51–9.57**: the pressure reflection passes to actual transport limits and forces them to vanish; differential reconstruction removes physical mass at or above the transport scale. The smaller scales are closed by Theorem 9.60. The $|\tau|^{4/7}$ rate remains a numerically supported balance | §9.16.8(h); Attachment C |
| **[EXACT]** | **Proposition 9.58, Lemma 9.59 and Theorem 9.60**: the cubic Dirac inverse controls every fixed $|\tau|^{-1/6}$ core; an exact six-component interval estimate closes the gap to the transport scale with controlled endpoint states. The full frozen-window lower bound tends to infinity. High-frequency compactness, full (C*) and the stated large-frequency target (S′a) are proved | §9.16.8(h); Attachment D |
| **[OPEN]** | Lemma H″: do admissible profiles with $M_0, M_1, M_2 > 0$ on $I$ exist? | §9.7 |

| | statement | status |
|---|---|---|
| **[COND-OA]** | every statement about the OpenAI construction, which has not yet been independently reviewed | §10 |

Symbolic and numerical verifications are reported where they were performed, with the tolerance.

### 1.3 Method: explicit decomposition rules

The reduction below is short because of a decomposition choice made before any calculation. That choice follows explicit rules for identifying which objects in a problem carry their own state, where a coupling is load-bearing, and what a projection discards [12]. The rules supply no equations, closures or conservation laws, and no conclusion depends on them: once the objects are fixed, the equations decide what is true. Where a reading suggested by the rules and the equations disagree, the equations win; §5.3 and §6.2(iii) are two places where such a statement is downgraded because it could not be proved.

The five rules used here are cited again at their points of use, so that each inference can be checked directly.

| rule | content | used at | what it does |
|---|---|---|---|
| **R1** | every apparently state-bearing object retains its own state until closure with another is demonstrated; the burden of proof is on **merging** | §5.1 | forces the exterior to be listed as a subsystem rather than deleted as "the far field" |
| **R9** | the **load-bearing test**: a coupling belongs to a subsystem's balance only if it is nonzero there, and not if it merely repairs compatibility elsewhere | §5.2 | given the source-free-core premise (A1b), classifies $\mathcal{T}[\tau]$ as a matching-problem term rather than a core term — **the classification the whole reduction rests on**; where the stress is supported is the premise, not an output of the rule |
| **R10** | no scalar closure where phase, sign, orientation or multiple stress components matter; the closure deficit is part of the state | §4(a), §9 | keeps $(\tau_{r\theta}, \tau_{rz})$ a pair, and makes realizability a **cone** condition rather than a sign |
| **R11** | observer-relative resolution must be represented, not turned into ontology | §8.2 | identifies $K$ as cut-defined and explains why it must cancel from any physical criterion |

| rule | content | used at | what it does |
|---|---|---|---|
| **R15** | separate first, merge only after closure is demonstrated — anti-flattening is **not** atomism | §5.1 | merges three of six candidates back down, the demanded stress being an exact functional of the other two |

The rules can be applied mechanically, and they constrain in both directions: R1 and R9 forbid merging objects that carry distinct state, and R15 forbids keeping apart objects that reconstruct each other. §5.1 applies R15 against a first decomposition that listed six subsystems where three close.

**Not used here.** The reformulation of blow-up as an event count, and the exact circulation cost $\Delta\Gamma = 4\pi^2\nu g$ per turnover of an axisymmetric core, belong to the same programme. Nothing below depends on them, and they are reported separately.

### 1.4 Provenance and use of AI tools

Two statements of this manuscript concern the order of events, so the order is given.

1. An earlier analysis of this architecture treated the interaction field as the driver of the core's collapse. It produced a numerical eigenvalue for the axial contraction rate and, from it, an exclusion of the architecture on energy grounds (§5.3).
2. That analysis, and several others in the same investigation, shared a recurring error: a coupling, a tensor component or a subsystem boundary had been collapsed into something simpler.
3. The rules of §1.3 codify earlier informal practice. They were written down as an explicit numbered list during this investigation, in response to step 2. The derivations, symbolic and numerical audits and drafting were carried out with the assistance of large language model tools, and rules that can be checked mechanically were needed to constrain that assistance. The author directed the work, checked the results and is responsible for the content.
4. The rules were then applied to the same problem. They reclassified the Reynolds stress as a matching-problem term by R9 and reduced the subsystem count from six to three by R15, overturning the earlier decomposition. The support statement this classification needs was later made explicit as the class premise (A1b) (§5.2).
5. The construction of §10 was examined.
6. The reduction of §§6–8 was derived after step 5 and then compared with that construction.

| claim | status |
|---|---|
| the rules were fixed before the problem | **no**: their principles were existing practice, but the numbered list was written during this investigation (step 3) |
| the rules were written to fit this result | **no**: once written, they contradicted the investigation's own earlier conclusion (step 4) |
| the rules contradicted the work that produced them | **yes**, and checkable: §5.3 against §5.2 |
| the exterior-exponent reduction predicted the construction of §10 | **no**: it postdates examination of that construction (steps 5–6) |
| the *architecture* and the $(K, G, \eta)$ ledger were reached independently of the construction | **yes**, in work dated 10–11 September 2026, before the construction was read; they are background here, not results of this manuscript |

| claim | status |
|---|---|
| priority | **none claimed, for anything** |

The methodological claim is therefore inductive, not predictive: applied mechanically, the rules reversed an earlier conclusion in the direction the mathematics then confirmed. §11.1 records two occasions on which they did not anticipate a result.

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## 2. The width law [EXACT]

Let $\rho(s)$ be the width of a strained axisymmetric core, defined by the second moment of its vorticity. Under an axial strain of rate $\sigma$ with radial diffusion,

$$\boxed{\frac{d\rho^2}{dt} = -2\sigma\rho^2 + 4\nu}$$

exactly — the two terms being compression and viscous spreading. Two consequences.

**(i) Contraction is a Reynolds-number condition.** Writing

$$K := \frac{\sigma\rho^2}{\nu},$$

W. gives $d\rho^2/dt < 0$ iff $K > 2$. Contraction is not a statement about how fast $\sigma$ grows but about the balance between strain and diffusion at the core scale.

**(ii) There is no universal parabolic floor on $\rho$.** A common inference — that viscosity forces $\rho^2 \gtrsim 2\nu s$ — follows from (W) only under a Type I hypothesis on $\sigma$. With $\sigma$ unconstrained, (W) permits $\rho^2 = c\,s$ for any $c > 0$, and then

$$-c = -2\sigma c\, s + 4\nu \qquad \Rightarrow \qquad \sigma = \frac{c + 4\nu}{2c\, s}, \qquad \boxed{K = 2 + \frac{c}{2\nu}}$$

is **constant in time** and exceeds 2. Parabolic radial scaling and a stationary $K$ are the same statement.

We use (W) twice below, and both times in the form

$$\boxed{\frac{\dot\rho}{\rho} = \frac{2\nu}{\rho^2} - \sigma = \sigma\Big(\frac{2}{K} - 1\Big).}$$

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## 3. The angular-momentum identity, and what supplies the swirl maximum principle [EXACT]

Let $m := ru^\theta$ be the angular momentum per unit mass of the **full** field. Then

$$\boxed{\partial_t m + u\cdot\nabla m = -\partial_\theta p + \nu\Delta m - \frac{2\nu}{r}\partial_\theta u^r}$$

identically. Two torques obstruct a maximum principle for $m$: an inviscid one, $-\partial_\theta p$, and a **viscous** one, $-2\nu r^{-1}\partial_\theta u^r$.

> **Consequence.** The swirl maximum principle
>
> [**E1**] $\quad \|\,\Gamma\,\|_{L^\infty}$ is non-increasing

> is supplied by exact axisymmetry of the **velocity field**, not by axisymmetry of the pressure. Axisymmetric pressure removes the first torque and leaves the second.

This is why (A1)'s interaction field is not decoration. It is the only thing in the architecture that can permit $\| \Gamma \|_\infty$ to grow, and §6.2 makes that precise. It is also a caution: [E1] is a property of a symmetry class and not of the equations, and using it outside that class is an error the literature on this problem invites.

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## 4. The mean equation, and the stress as an exact flux divergence [EXACT]

Averaging the $\theta$-momentum equation azimuthally and writing $\bar{\Gamma} = r\bar{u}^\theta$:

$$\boxed{\partial_t \bar{\Gamma} + \bar{u} \cdot \nabla \bar{\Gamma} = \nu \left( \partial_r^2 - \frac{1}{r}\partial_r + \partial_z^2 \right) \bar{\Gamma} - \mathcal{T}[\tau] + r\bar{f}^\theta}$$

with

$$\mathcal{T}[\tau] = \frac{1}{r}\partial_r \left( r^2 \tau_{r\theta} \right) + r\, \partial_z \tau_{z\theta}.$$

$\mathcal{T}$ is a **flux divergence**: the interaction field redistributes angular momentum and creates none. Equivalently, in the $u^\theta$ equation the stress enters as

$$\frac{\mathcal{T}[\tau]}{r} = \left( \partial_r + \frac{2}{r} \right) \tau_{r\theta} + \partial_z \tau_{z\theta}.$$

Two remarks that matter downstream.

**(a) The stress is a pair, not a scalar [R10].** Both $\tau_{r\theta}$ and $\tau_{rz}$ are dynamically active — the first in (S′), the second in the axial momentum balance. Any reduction that represents the interaction field by one number has discarded a component that acts on the mean.

**(b) $\mathcal{T}$ is defined everywhere but need not be nonzero everywhere.** Where $\mathcal{T}$ is supported is not fixed by (S′). §5 determines it.

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## 5. The decomposition

### 5.1 Candidate subsystems, and the merge test

**[R1]** list every apparently state-bearing object before deciding which belong together, with the burden of proof on merging. For this architecture the candidates are

$\mathcal{E}_0$ (exterior), $\mathcal{E}_1$ (core), $\mathcal{E}_2$ (matching layer), $\mathcal{E}_3$ (interaction field), $T$ (demanded stress), $\mathcal{K}$ (realizability condition).

**[R15]** then requires the merge test — keep two objects separate only if neither reconstructs the other without loss. Anti-flattening is not atomism, and this table is where the protocol constrains us in the restrictive direction: our own first decomposition of this architecture listed all six as subsystems, and three of them close.

| candidate | carries state? | reconstructible? | verdict |
|---|---|---|---|
| $\mathcal{E}_0$ exterior | the profile $\Gamma_{\text{ext}}$, i.e. the number $\alpha$ | **no** — the core cannot produce it | **subsystem** |
| $\mathcal{E}_1$ core | $(\rho, \ell, \Lambda, K)$ and the inner profile | no | **subsystem** |

| candidate | carries state? | reconstructible? | verdict |
|---|---|---|---|
| $\mathcal{E}_2$ matching layer | the residual of joining $\mathcal{E}_1$ to $\mathcal{E}_0$ | **yes** — a functional of $(\mathcal{E}_0, \mathcal{E}_1)$ | **merge: a region** |
| $\mathcal{E}_3$ interaction field | amplitudes, relative phases, wavevector orientation, with its own growth–decay dynamics; not a functional of $\bar{u}$ | **no** | **subsystem** |
| $T$ | — it is *determined*: $T_{r\theta} = -r^{-2}\int_0^r s^2 R_\theta \, ds$ | yes | **merge: a relation** |
| $\mathcal{K}$ | — it is a *condition*, not a state | yes | **merge: a condition** |

**Three subsystems $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_3)$ and two determined relations $(T,\ \mathcal{K})$.**

## 5.2 The load-bearing test classifies the stress [R9, given (A1b)]

A coupling term belongs to the governing balance of a subsystem only if it is **nonzero** there — and not if it is merely repairing a compatibility failure somewhere else.

**What the rule can and cannot do.** R9 is a classification rule. *Given* where $\mathcal{T}[\tau]$ is nonzero, it says to which subsystem's balance the term belongs. It cannot establish where $\mathcal{T}[\tau]$ is nonzero: that is a dynamical fact, and (S′) holds for any $\tau$. An earlier version of this analysis let the rule supply it, arguing that the interior of a self-similar core "is a solution of its own problem" — which presupposes the conclusion. That argument is withdrawn.

**Route A — deriving the support — is not available at this level.** The leading-order stress is determined by the profile through (4.10)–(4.11), and whether it vanishes on a given radial interval is a property of the profile, not an identity of the equations. §9 makes this concrete twice: in the reflection-symmetric class the demand cannot vanish throughout the region inboard of the layer (Theorem 9.18), and in the class $\partial_\eta E|_0 = 0$ it must be nonzero across the whole passage of the circulation slope through zero (§9.16.9).

**Route B is therefore the correct one.** The support statement is the class premise (A1b), and with it R9 performs the classification:

$$\mathcal{T}[\tau] = 0 \ \text{ on } \mathcal{E}_1 \ \text{(A1b)}, \qquad \mathcal{T}[\tau] \neq 0 \ \text{ only on } \mathcal{E}_2 \ \text{(R9)}.$$

Within (A1b) the coupling graph is a **chain**, not a loop:

$$\mathcal{E}_0 \xrightarrow[\Gamma_{\text{ext}}(\rho)]{\text{boundary datum}} \mathcal{E}_1 \xrightarrow[T]{\text{residual}} \mathcal{E}_2 \xrightarrow[\mathcal{K}]{\text{cone}} \mathcal{E}_3.$$

Nothing returns to the core at leading order. There is no gain to price.

## 5.3 Decomposition in one paragraph

The alternative decomposition is the natural one, and it is wrong. Treat the interaction field as the *driver* of the core's collapse: then $\tau$ is load-bearing inside $\mathcal{E}_1$, its axial profile must be matched to the swirl's, the growth of $\Lambda$ must be paid for by a stress correlation and a saturation amplitude, and the axial contraction rate $\eta$ becomes the eigenvalue of a two-point boundary-value problem. Carried out, that programme returns $\eta > 0$ and the architecture appears **excluded on energy grounds**. Every step of it is correct; the exclusion is not a statement about this class, because it excludes a core driven by an internal stress and, in the class (A1b), the stress is not there. Given (A1b), the load-bearing test reclassifies one term, and the eigenvalue

problem, the correlation, the saturation law and the exclusion all disappear together — $\eta$ becoming kinematic and negative. That is the entire content of the rules of §1.3, and it is the reason the following two sections are four lines long each. **Note which way the correction ran.** The core-driven reduction was an earlier analysis, made before the rules were written down; what removed it was rule R9 applied to that analysis — once the premise that the core is source-free had been stated, which R9 cannot supply by itself (§5.2), and which §9.16.9(h) shows is realizable on the midplane.

---

## 6. The supply theorem

### 6.1 The core cannot make swirl [EXACT | A1b]

By (A1b), in the form (5.1), and (A1), on $\mathcal{E}_1$ the mean swirl equation (S′) reduces to

$$\partial_t \bar{\Gamma} + \bar{u} \cdot \nabla \bar{\Gamma} = \nu \left( \partial_r^2 - \frac{1}{r} \partial_r + \partial_z^2 \right) \bar{\Gamma}$$

— pure advection and diffusion, with **no source**. The operator on the right admits a maximum principle for $\bar{\Gamma}$ (this is [E1] restricted to the core, where the field is axisymmetric), so $\bar{\Gamma}$ attains no interior maximum in excess of its parabolic boundary values. With (A2) and (A3),

$$\boxed{\Lambda(s) \;=\; \| \bar{\Gamma} \|_{L^\infty(C_s)} \;\le\; \max_{\partial_{\mathrm{par}} C_s} \bar{\Gamma} \;\asymp\; \Gamma_{\mathrm{ext}}\big(\rho(s)\big) \;\asymp\; \rho(s)^{-\alpha}.}$$

> **The swirl does not grow; the core shrinks into a steeper ambient profile and samples ever larger values of it.**
>
> Note the direction: (6.1) is an **upper bound**, and it is one-sided. Every statement below that needs $\Lambda$ to be as large as $\Gamma_{\mathrm{ext}}(\rho)$ — rather than merely no larger — invokes **(A4)**, and is labelled accordingly. (A4) is a genuine restriction: a core whose swirl decays relative to the ambient profile satisfies (6.1) and none of §§6.2(ii), 7, 8.

This replaces a model by a theorem. No correlation coefficient and no saturation amplitude appear, and none is needed.

### 6.2 Where non-axisymmetry is and is not localized

Three statements of decreasing strength, with decreasing status: the first is exact, the second is conditional, the third is not proved at all.

**(i) [EXACT].** $\mathcal{E}_1$ cannot source a growing interior maximum of $\bar{\Gamma}$; the growth of $\Lambda$ arrives through the core boundary. This is §6.1.

**(ii) [COND | GROWING-SUPPLY], not [EXACT].** Two conditions are needed and both must be stated: $\alpha > 0$, and **(A4)**. Given them, $\Lambda \asymp \rho^{-\alpha} \to \infty$, and since $\Lambda \le \| \Gamma \|_{L^\infty}$ the **global** $\| \Gamma \|_\infty$ must grow, which [E1] forbids for a globally axisymmetric velocity field. Hence

| if $\alpha > 0$ and the supply is saturated, non-axisymmetry is necessary and cannot be confined to the source-free core. |
|---|

> **Why the conditions are not cosmetic.** (6.1) alone gives an upper bound on $\Lambda$ and therefore no lower bound on $\| \Gamma \|_\infty$ ; without (A4) the global maximum need not grow at all, and a globally axisymmetric field is not excluded. Without $\alpha > 0$ it does not grow even with (A4). **The necessity of non-axisymmetry is a conditional result about a growing architecture, not a theorem about Navier–Stokes.** Statement (i) is unconditional; this one is not, and an earlier version of this analysis labelled it [EXACT] in the claim table, which was wrong.

**(iii) Not proved.** That the non-axisymmetric stress is supported in the matching annulus *specifically*. Statement (ii) excludes the core interior and requires the supply to arrive through the boundary; it does not exclude a non-static exterior, a stress spread over a range of radii, or transport of the global maximum from elsewhere. The annulus is the natural location and, in the one construction known (§10), the actual one. It is not a theorem here and is not used as one.

---

## 7. The reduction

Throughout, $\sigma$ is the axial strain at the core, $K = \sigma\rho^2/\nu$, and

$$G := \frac{\dot{\Lambda}/\Lambda}{\sigma}, \qquad \eta := \frac{\dot{\ell}/\ell}{\sigma}$$

are the swirl growth rate and the axial contraction rate measured on the strain clock.

### 7.1 The swirl growth rate [EXACT | SUPPLY]

Differentiate (6.1) **as an equality**, which is (A4), and substitute (2.2):

$$\frac{\dot{\Lambda}}{\Lambda} = -\alpha\frac{\dot{\rho}}{\rho} = -\alpha\,\sigma\left(\frac{2}{K} - 1\right) \qquad \Rightarrow \qquad \boxed{G = \alpha\left(1 - \frac{2}{K}\right).}$$

*Verified symbolically.* The swirl growth rate is the exterior exponent times the non-viscous fraction of the width law. It contains no information about the interaction field.

### 7.2 The axial contraction rate [EXACT | AXIAL-BALANCE]

One further hypothesis, named because the result depends on it:

> **(A5) Axial balance.** Axial transport remains in the leading balance, $|u^z|/\ell \asymp 1/s$, with $|u^z| \asymp |u^\theta|$.

Since $\Lambda = \rho\, u^\theta$ and $\Lambda \asymp \rho^{-\alpha}$ with $\rho \asymp s^{1/2}$, the velocity exponent is $A = \frac{1}{2} + \frac{\alpha}{2}$; (A5) then forces the length exponent $D = 1 - A = \frac{1}{2} - \frac{\alpha}{2}$, i.e. $\ell \asymp s^{1/2\,-\,\alpha/2}$, and

$$\boxed{\eta = -(1-\alpha)\left(1 - \frac{2}{K}\right) \;<\; 0 \qquad \text{for } 0 < \alpha < 1,\ K > 2.}$$

*Verified symbolically.*

> **The axial sign is kinematic.** It is not the eigenvalue of a boundary-value problem, and it does not require a model of the stress profile. It is negative for the whole admissible range of $\alpha$.

### 7.3 Three relations, and one that becomes an identity

From (7.1) and (7.2), with $K > 2$:

$$G - \eta = \frac{K-2}{K} \quad \text{(independent of } \alpha\text{)}, \qquad 1 + \eta - G = \frac{2}{K}, \qquad \boxed{a := \frac{G}{2(G-\eta)} = \frac{\alpha}{2}}$$

where $a$ is the swirl exponent, $\Lambda \asymp s^{-a}$. Consequently the scalar evolution equation for the collapse Reynolds number,

$$\frac{dK}{dT_\sigma} = K\left(G - 1 - \eta\right) + 2, \qquad dT_\sigma = \sigma\, dt,$$

has $K(1 + \eta - G) = 2$ and therefore vanishing right-hand side **for every** $\alpha$ **and every** $K > 2$.

> **(K3) is [EXACT | AXIAL-BALANCE], and carries no information about the interaction field.** Its content is the exact width law together with (A5), and nothing else. A stationary $K$ is not a dynamical selection; it is what (W) and (A5) say. This is worth stating because (K3) is easily mistaken for a closure relation whose satisfaction is evidence about the fluctuation. It is not.

---

# 8. The window

## 8.1 Growth and energy

**Contraction** is $K > 2$, from (W), and is independent of $\alpha$: a core may shrink at $\alpha = 0$.

**Swirl growth** requires $G > 0$, hence

$$\boxed{\text{within } K > 2: \qquad \Lambda \text{ grows} \Leftrightarrow \alpha > 0.}$$

At $\alpha = 0$ — constant ambient circulation, a line vortex — the core contracts with $\Lambda$ constant and nothing concentrates.

**Energy.** The core contribution is $E_{\text{core}} \asymp \Lambda^2 \ell \asymp s^{-2\alpha}\, s^{1/2\,-\,\alpha/2} = s^{1/2\,-\,3\alpha/2}$, so $E_{\text{core}} \to 0$ iff $\alpha < \frac{1}{3}$. Equivalently, in the $(G, \eta)$ variables the condition is $\eta < -2G$, and

$$-2G - \eta = \frac{K-2}{K}\big(1 - 3\alpha\big) \qquad \Rightarrow \qquad \boxed{E_{\text{core}} \to 0 \Leftrightarrow \alpha < \tfrac{1}{3}.}$$

*Verified symbolically, and checked at sixteen $(\alpha, K)$ pairs.* Hence the window

$$\boxed{0 < \alpha < \tfrac{1}{3}, \qquad \text{equivalently} \qquad 0 < a < \tfrac{1}{6}.}$$

**Scope.** (8.1) is the condition for the **core contribution** to the energy to vanish, within a single-parameter self-similar family satisfying (A1)–(A3) and (A5). It is not a universal finite-energy criterion for Navier–Stokes.

## 8.2 $K$ cancels, and it had to [R11]

The factor $(K-2)/K > 0$ divides out of (8.1) exactly. This is not a convenience.

$K = \sigma\rho^2/\nu$ depends on $\rho$, and $\rho$ is the radius at which one chooses to draw the core boundary. Where that choice is explicit — a level set of a similarity variable, say — $K$ is a function of the cut.

$$\boxed{\text{A physical criterion may not depend on a cut-defined } K.}$$

Two honest restrictions. First, this does **not** generalize to every $K$: a construction that defines $\rho$ intrinsically, by a second moment of vorticity or a half-maximum radius, has an intrinsic $K$, and the objection does not apply to it. Second, the cancellation is **not** a general invariance of the equations: $G$ and $\eta$ carry the same factor $(1 - 2/K)$ because both $\Lambda$ and $\ell$ are powers of $\rho$, which is what a single-parameter self-similar family means. Within that family the cancellation is decisive; outside it, it is not claimed.

> **A diagnostic, in passing.** A criterion built on a cut-defined quantity will move when the cut moves. If successive attempts at a threshold for this architecture keep shifting, that is the symptom, and (8.1) is the cure: every condition above is a statement about $\alpha$, which is a property of a field.

---

## 9. Realizability: what the window does not contain

The chain of §5.2 closes iff the stress that the matching relation **demands** is one the interaction field can **supply**. Half of that question is now settled; the other half is not.

### 9.1 The demand [EXACT]

Requiring the residual of the mean to be cancelled by a flux divergence fixes

$$T_{r\theta}(r) = -\frac{1}{r^2}\int_0^r \sigma^2 R_\theta \, d\sigma, \qquad T_{rz}(r) = -\frac{1}{r}\int_0^r \sigma R_z \, d\sigma,$$

the integration constants being fixed by regularity on the axis. For $T$ to vanish outside the matching layer, so that the core and the exterior are left undisturbed, two moment conditions must hold:

$$\int_0^\infty r^2 R_\theta \, dr = 0, \qquad \int_0^\infty r\, R_z \, dr = 0.$$

### 9.2 The supply is a circular cone whose axis is the radial shear [EXACT]

A zero-mean oscillatory field of given local orientation produces a covariance vector $v = (\langle u_r' u_\theta' \rangle, \langle u_r' u_z' \rangle)$; two independent families span a cone, and realizability is $T \in \mathrm{int}\mathcal{C}$. That cone can be computed in closed form. Writing $s$ for the normalised radial shear and $p_s$ for the integrated inviscid-stress vector, so that the demand is $T = F(p_s - s)$ with $F > 0$,

$$\boxed{\mathcal{C} = \left\{T:\ T\cdot s > 0,\ \angle(T,s) < \theta_c\right\}, \qquad \theta_c = \arctan\sqrt{\frac{2a}{|s|^2 - 2a}}, \qquad a = s_1 > 0,}$$

non-degenerate exactly when $|s - e_1| > 1$. The change of variables that diagonalises the condition is **conformal**, which is why the cone is circular; and **its axis is the shear itself**. Two consequences matter.

- **The shear appears on both sides.** It sets the cone *and*, through $T = F(p_s - s)$, the demand. Any argument that treats the background as fixed while asking whether some fluctuation has the right sign has already lost the problem — this is [R10] and [R9] acting together.
- **Steering is possible.** A profile modulation that is $O(N^{-1})$ in value but order one in radial derivative moves $s$, hence the cone, while leaving the integrated $p_s$ and therefore $T$ almost fixed.

### 9.3 Pointwise realizability is solved, and is not an obstruction [EXACT]

> **Theorem 9.1.** Given $p_s$, an admissible shear exists **iff** $p_{s,2} \neq 0$ or $p_{s,1} > 2$.
>
> **Theorem 9.2.** A shear placing the demand on the cone axis — maximal aperture margin — exists **iff**
>
> $$\Theta := |p_s|^2 - 2p_{s,1} = |p_s - e_1|^2 - 1 > 0,$$
>
> and then $v_s := |s|^2/a$ may be taken anywhere in $\left(2,\ 2 + \Theta/p_{s,1}\right)$.

Both are proved by reducing admissibility to $\exists f \in (-\frac{\pi}{2}, \frac{\pi}{2})$: $(p_{s,1} - 2)\cos f + p_{s,2}\sin f > 0$, the aperture inequality being non-binding near the lower endpoint of the admissible radius range. So the **same** unit circle centred at $(1, 0)$ governs both: $s$ must lie outside it for the cone to exist, $p_s$ outside it for the demand to be placeable on the axis.

> **The cone half of the realizability problem therefore obstructs nothing.** Infeasibility is confined to the measure-zero ray $\{p_{s,2} = 0,\ p_{s,1} \leq 2\}$. A conjecture that $\Theta > 0$ is also *necessary* for a uniform margin is **false** — at $p_s = (0.5, 0.2)$, where $\Theta = -0.71$, off-axis shears still reach $v_s \approx 2.22$ — so only the on-axis statement of Theorem 9.2 is claimed.

### 9.4 The cone parameter is the exterior exponent [EXACT]

The shear $s = (a, -b_s)$ can be computed rather than posited. With $D_X = X\partial_X$ the logarithmic radial derivative and $H$ the angular-momentum profile, the construction's shear parameter is $a = 2 - 2D_X \log H$; and since $X \propto r^2$ and $\Gamma \propto H$ at fixed axial similarity coordinate,

$$D_X \log H = \frac{1}{2}\frac{d\log\Gamma}{d\log r} = -\frac{1}{2}\alpha_{\mathrm{loc}}(r), \qquad \boxed{a = 2 + \alpha_{\mathrm{loc}}(r), \qquad b_s = \frac{2D_X U}{E}}$$

where $\alpha_{\mathrm{loc}}(r) := -d\log\Gamma/d\log r$ is the **local** circulation exponent of §1.1(A3) and $b_s$ is the normalised axial shear. Hence, verified symbolically,

$$v_s - 2 = \frac{\alpha_{\mathrm{loc}}(2 + \alpha_{\mathrm{loc}}) + b_s^2}{2 + \alpha_{\mathrm{loc}}}, \qquad \theta_c = \arctan\sqrt{\frac{2(2 + \alpha_{\mathrm{loc}})}{\alpha_{\mathrm{loc}}(2 + \alpha_{\mathrm{loc}}) + b_s^2}}.$$

> **Theorem 9.3.** The cone is non-degenerate iff $\alpha_{\mathrm{loc}}(2 + \alpha_{\mathrm{loc}}) + b_s^2 > 0$, for which $\alpha_{\mathrm{loc}} > 0$ is sufficient, and necessary where the axial shear vanishes. In the exterior, where $U = 0$ and $\alpha_{\mathrm{loc}} = \alpha$, this reads
>
> $$s = (2 + \alpha,\, 0), \qquad v_s - 2 = \alpha, \qquad \theta_c = \arctan\sqrt{2/\alpha}.$$

Two consequences, and the first is the reason this section exists.

**(i) The cone condition and the window condition are one condition.** §8 derived $\alpha > 0$ kinematically, as what makes the shrinking core sample a rising ambient circulation. The cone condition $v_s > 2$ is derived from a covariance constraint on what an oscillatory field can supply. In the exterior they are the **same inequality**, and in general the first implies the second:

$$\boxed{\alpha > 0 \;\Leftrightarrow\; \text{the exterior covariance cone is nondegenerate,} \qquad \alpha_{\mathrm{loc}} > 0 \;\Rightarrow\; \text{the local cone is nondegenerate.}}$$

**This is cone *existence*, not realizability.** It says the wave family spans a nondegenerate cone at that radius. Full realizability additionally requires the demanded direction $w = p_s - s$ to lie **inside** it, which is a separate condition on $p_s$ and is the subject of §9.6. An earlier version of this section wrote the implication as "the interaction field can supply the stress at all", which overstates it.

Where $\alpha_{\mathrm{loc}} < 0$ the axial shear may still rescue admissibility, through $b_s^2 > |\alpha_{\mathrm{loc}}|(2 + \alpha_{\mathrm{loc}})$; where $b_s = 0$ it cannot, and then $a > 0$ with a non-degenerate cone forces $d\Gamma/dr < 0$ — **the matching layer must lie outside the angular-momentum maximum**, which is the centrifugally unstable region where such oscillations are amplified. The cone algebra and the amplification mechanism demand the same geometry.

**(ii) The cone does not degenerate at the top of the window.** On the exterior side,

| $\alpha$ | 0.002 | 0.02 | 0.1 | 0.2 | 1/3 |
|---|---|---|---|---|---|
| $v_s - 2$ | 0.002 | 0.02 | 0.1 | 0.2 | 1/3 |
| $\theta_c$ | 88.2° | 84.3° | 77.4° | 72.5° | 67.8° |

so as $\alpha \to \frac{1}{3}$ the margin reaches $\frac{1}{3}$ and the aperture is still 67.8°. **The cone degenerates at the opposite end**, $\alpha \to 0$, where $v_s - 2 \to 0$. Whatever bounds $\mathcal{A}$ above, it is not the cone.

### 9.5 Uniformity [EXACT]

A pointwise strict inequality need not give a uniform gap on a noncompact domain. Here the two regions separate.

- On the matching interval $I = [X_a, X_b] \Subset (0, \infty)$, which is **compact**, $\alpha_{\mathrm{loc}}$ and $b_s$ are continuous, so pointwise positivity gives a uniform gap directly.

- On the noncompact exterior tail, $v_s - 2 = \alpha$ **identically in the radius** — a constant, not merely a positive function — and there the demanded stress vanishes, so the angular condition is vacuous.

> **The uniform-margin problem is therefore structurally closed**, and reduces to pointwise positivity on a compact interval. What is not closed is that pointwise condition in the **interior** of the matching layer.

### 9.6 Structure theorem: the radial state is not the profile state [EXACT]

The margins can be written exactly. With $a = 2 + \alpha_{\mathrm{loc}}$, $s = (a, -b_s)$, $s^{\perp} = (b_s, a)$ and $w = p_s - s$:

$$M_0 = |s|^2 - 2a = \alpha_{\mathrm{loc}}^2 + 2\alpha_{\mathrm{loc}} + b_s^2, \qquad M_1 = w \cdot s = (2 + \alpha_{\mathrm{loc}})w_1 - b_s w_2, \qquad M_2 = 2a(w \cdot s)^2 - M_0(w \cdot s^{\perp})^2,$$

and realizability at radius $X$ is $M_0, M_1, M_2 > 0$. The natural hope is that $\{M_i > 0\}$ is a forward-invariant region for a radial ODE, so that realizability would follow from edge data. **It is not, and cannot be** — and the reason is a positive statement about the problem rather than a defect of the attempt.

> **Theorem 9.4 (the demanded stress obeys a driven linear system).** Writing $w_1 = p_{s,1} - a$, $w_2 = p_{s,2} + b_s$, the construction's own stress equations are equivalent to
>
> $$D_X w_1 + l\, w_1 = \Sigma_q, \qquad D_X\left(E w_2\right) = \Sigma_n,$$
>
> where $\Sigma_q, \Sigma_n$ are the **zero-stress defects** — they vanish identically precisely when the profile solves the vanishing-residual equations. Verified symbolically.

Three consequences, and together they change what kind of problem this is.

**(i) The system is not closed in any finite radial state.** The sources driving $\Sigma_q, \Sigma_n$ contain $\partial_\eta \log E$, $\partial_\eta U$, $\partial_\eta \Pi$ and $\partial_\eta$ of the radial average of $U$. These are **axial** derivatives; no finite-dimensional state in the radial variable reconstructs them. The realizability problem is a PDE in $(X, \eta)$, not an ODE in $X$.

**(ii) The profile is a design variable, not a trajectory.** $Q_s$ and $N_s$ are *determined* by $(E, U, \Pi)$ through their own radial equations with the integration constants fixed by axis regularity. So every margin is a **functional** $M_i[E, U](X, \eta; \alpha)$ of freely chosen profiles, constrained only by axis regularity, exterior matching and the moment conditions.

**(iii) Hence no invariant-region proof exists, and none is needed.** A linear equation with free forcing can be steered anywhere: $\mathcal{R} = \{M_0, M_1, M_2 > 0\}$ is neither forward-invariant nor forward-repelling. The question is not whether a trajectory escapes but whether finitely many global constraints leave enough freedom in an infinite-dimensional design space.

**radial state ≠ full profile state.**

That is the content of (i) stated invariantly. A finite radial state is a **projection** of the profile, and the realizability margins are not functions of that projection: they depend on axial derivatives the projection discards. The attempted invariant-region proof is what exposed it — the proof could not close because there was nothing to close, and the obstruction is informative rather than technical.

Lemma H is a constrained design problem: infinitely many degrees of freedom, finitely many constraints.

> **Methodological note, because it is the same lesson twice.** §5 used the load-bearing test to remove a false problem — the core paying for a stress that vanishes there. This section nearly replaced the surviving problem with a different false one: a finite radial state standing in for a profile. The second flattening was caught by attempting the proof rather than by inspection, which is the case the decomposition rules are weakest at catching in advance and strongest at diagnosing afterwards.

**The edge data, correctly interpreted.** At either edge the stress vanishes, so $w \to 0$ and $M_1, M_2 \to 0$: the condition must be read through the leading factor. At a simple zero $X_e$, Theorem 9.4 gives $w_1 \sim \Sigma_q(X_e)\log(X/X_e)$ and $w_2 \sim [\Sigma_n(X_e)/E(X_e)]\log(X/X_e)$, so the limiting direction is

$$\widehat{w}(X_e) = \left(\Sigma_q,\ \Sigma_n/E\right)\big|_{X_e},$$

and the edge condition is that $\widehat{w}$ lie in the **closed** cone. **The edge datum is the zero-stress defect, not the stress.**

### 9.7 Status of Lemma H

| **Lemma H″.** Do there exist profiles $(E, U)$ meeting axis regularity, exterior matching and the moment conditions, with $M_0, M_1, M_2 > 0$ on $I$? |
|---|

**Where this stops, exactly.** Nothing here exhibits such profiles, and nothing here forbids them. What has been removed is the hope of settling the question by a trajectory argument, and the associated worry that the margins might be dragged out of $\mathcal{R}$ by dynamics they do not have.

**What would settle it.** Either a construction — which is what OA supplies at one value of $\alpha$, **[COND-OA]** — or a proof that the finitely many constraints are incompatible with the pointwise inequalities. By Theorems 9.1–9.2 the latter cannot come from the cone alone, since the pointwise infeasible set is a measure-zero ray. §9.15 bears on this: the *tangent* problem at a fixed background is feasible throughout a tested family **[NUM]**. Were that feasibility established analytically with a strict uniform margin, the higher-order terms would be perturbative and an incompatibility proof could not come from the cone at any order; with the present numerical status that consequence is conditional.

**No upper obstruction has been found.** §9.4 shows the exterior cone *improves* as $\alpha \to \frac{1}{3}$, so any bound on $\mathcal{A}$ must originate in the matching layer; and §9.6 shows the matching layer imposes no dynamical constraint at all. Whether $\mathcal{A} = (0, \frac{1}{3})$ remains open.

### 9.8 The function-space formulation, and one theorem in it

Since the problem is a design problem, the next step belongs in the profile space rather than along a radius. State it precisely.

Let $\mathcal{P}$ be the set of profiles $(E, U)$ on the closed matching interval that are admissible in the sense of §1.1 — axis-regular, joined to the prescribed exterior with exponent $\alpha$ — taken in a Banach space $\mathcal{X}$ with $C^1$ control of the radial log-derivatives. Let

$$\mathcal{M}: \mathcal{P} \to \mathbb{R}^m, \qquad \mathcal{U} := \left\{\Phi \in \mathcal{P}:\ M_0, M_1, M_2 > 0 \text{ on } I\right\},$$

with $m = 2$ at the level this paper needs. Then

$$\boxed{\text{Lemma H}'' \Leftrightarrow \mathcal{M}^{-1}(0) \cap \mathcal{U} \neq \varnothing.}$$

**The two ingredients this formulation needs.** (a) $\mathcal{U}$ is **open** — true, and worth checking, because the margins involve first radial derivatives: $M_0$ through $a$ and $b_s$, and $M_1, M_2$ through $w$, whose components are radial primitives of sources built from first derivatives. So all three margins are $C^1$-continuous and $\mathcal{U}$ is open in $\mathcal{X}$. (b) $\mathcal{M}$ is a **submersion** on $\mathcal{U}$, i.e. $D_\Phi \mathcal{M}$ is onto $\mathbb{R}^m$ using directions supported where the margin is not thin. **(b) is open.** With it, $\mathcal{M}^{-1}(0)$ is a closed submanifold of codimension $m$ meeting $\mathcal{U}$ near any near-solution, and Lemma H″ follows from a base point with small $|\mathcal{M}|$.

> **Theorem 9.5 (repair versus margin) [EXACT as a scaling].** Let the moment discrepancy be $\mu = |\mathcal{M}[\Phi_0]|$, let the cone margin of $\Phi_0$ be $\delta_0 > 0$, and let the repair be supported in a region of width $W$. A correction built from fixed-shape bumps of amplitude $\varepsilon$ and width $\delta \leq W$ changes the moments

> by $O(\varepsilon\delta)$ and the margins by $O(\varepsilon/\delta)$, so repairing $\mu$ costs $O(\mu/\delta^2)$ in $C^1$. Hence the repair preserves the margin provided
>
> $$\boxed{\mu \ll \delta_0 W^2.}$$
>
> The implied constant depends only on the bump shape and the moment weights.

Two readings, and the second is the useful one.

- **The asymmetry is favourable.** Margins are $C^1$-continuous; moments are integrals and respond at $C^0$ scale times the width. That is why a repair can be made cheap in the norm that matters, and it is why *wide, low* corrections are the right instrument rather than narrow ones.
- **The bump machinery is sufficient and is not delicate.** A modulation leaving discrepancies $\mu = O(1/N)$ against a fixed margin and width satisfies $\mu/\delta_0 W^2 = O(1/N) \to 0$, so some finite $N$ works, with no further structure required. What Theorem 9.5 does **not** say is that such machinery is *necessary*: any correction family meeting the inequality will do, and the inequality is the whole requirement.

**So the remaining question is surjectivity, and only surjectivity.** §9.9 answers it.

### 9.9 The moment map is a submersion [COND on the transcription of the source equations]

**The target is a function space, not $\mathbb{R}^m$.** The moment identities must hold as functions of the axial similarity coordinate $\eta$, so $\mathcal{M}\colon \mathcal{P} \to C([-1,1])^2$. Surjectivity is therefore a **fibrewise** question in $\eta$, together with uniformity — and since perturbations $\delta E, \delta U$ are functions of $(X,\eta)$, each fibre carries its own independent pair of bump directions. At each $\eta$ the question is a $2\times 2$ determinant.

**The functionals.** With the integration constants fixed by axis regularity, the stress coefficients are radial primitives, and their exterior tails vanish exactly when the total integrals do:

$$c_Q(\eta) = \int_0^\infty H\, S_q \, dX, \qquad c_N(\eta) = \int_0^\infty S_n \, dX.$$

**The weights.** Take $\delta E$ and $\delta U$ compactly supported in the repair region. Since $S_n$ contains $-WD_X U$ and $S_q$ contains $-Wl$ with $\delta l = D_X(\delta E/E)$, one integration by parts in each — boundary terms vanishing by compact support, verified numerically to $1.7\times 10^{-17}$ — gives

$$\delta c_N|_{\delta U} = \int \Omega\, \delta U \, dX, \quad \Omega := \partial_X\big(XW\big); \qquad \delta c_Q|_{\delta E} = \int \Omega_Q \frac{\delta E}{E}\, dX, \quad \Omega_Q := \partial_X\big(XHW\big).$$

On the outer collar, where $W \to 1$ and $H \sim \sqrt{2}\, c_\infty X^{-h}$, these are $\Omega \approx 1$ and $\Omega_Q \approx \sqrt{2}\, c_\infty (1-h) X^{-h}$ — **both of one sign and bounded away from zero.**

**The cross term is small, and vanishes on the midplane.** Where $U = 0$, $\delta E$ reaches $S_n$ only through the pressure. Integrating the transport term by parts, the two pressure terms combine with coefficient $4A - 2 = 2\alpha$, so

$$\delta c_N|_{\delta E} = 2\alpha\, \eta \int \delta\Pi \, dX - \int d\,(\delta\Pi)_\eta \, dX.$$

Both terms carry $\eta$ or an $\eta$-derivative. Hence

> **Theorem 9.6 (submersion).** At $\eta = 0$ the fibrewise Jacobian is **exactly triangular**, with diagonal entries $\Omega_Q$- and $\Omega$-weighted and therefore nonzero:
>
> $$\det J(0) = \Big(\int \Omega_Q \tfrac{\delta E}{E}\Big)\Big(\int \Omega\, \delta U\Big) \neq 0.$$

> For $\eta \neq 0$ the lower-left entry is $O(\alpha|\eta|)$, so $\det J(\eta)$ is bounded away from zero uniformly on $|\eta| \leq 1$ once $\alpha$ is below an explicit threshold fixed by the profile bounds. $D_\Phi \mathcal{M}$ **is then a submersion with bounded right inverse**, using directions supported in the repair region.

**The degeneracy test, in general.** Without the collar assumption, placing one bump in $E$ at $X_1$ and one in $U$ at $X_2$ and varying both, $\det J \equiv 0$ requires

$$\frac{\omega_Q^E}{\omega_N^E} \equiv \frac{\omega_Q^U}{\omega_N^U} \equiv \text{const on the repair region,}$$

two function identities at once. **Surjectivity is the generic case**, and the collar computation exhibits an explicit non-degenerate regime rather than relying on genericity.

> **Status, stated precisely.** The by-parts weights, the coefficient $4A - 2 = 2\alpha$, the triangularity at $\eta = 0$ and the degeneracy test are this paper's, and are verified. They rest on the **transcription** of the source terms $S_q, S_n$ from the primary construction, which is **[COND-OA]**: if those sources carry a term this reading missed, the weights change and Theorem 9.6 must be recomputed. The structure — fibred over $\eta$, triangular on the midplane, degenerate only under two function identities — does not depend on the coefficients.

### 9.10 What this leaves

Combining: $\mathcal{U}$ is open (§9.8); $\mathcal{M}$ is a submersion on it (Theorem 9.6); and a moment discrepancy $\mu$ can be repaired without eating a margin $\delta_0$ provided $\mu \ll \delta_0 W^2$ (Theorem 9.5). Hence

| If some $\Phi_0 \in \mathcal{U}$ has moment discrepancy $\mu \ll \delta_0 W^2$, then Lemma H″ holds at Level A. |
|---|

**The whole remaining question is the existence of one near-solution in the cone-admissible set.** That is a strictly smaller statement than Lemma H″: it asks for an approximate solution with a margin, not an exact one, and the correction is then automatic. It is also exactly what a construction supplies, which is why OA's existence at one $\alpha$ is evidence of the right kind — **[COND-OA]**, §10.

No obstruction has been found at any stage: not in the cone (§9.3), not in the exterior margin (§9.4), not in uniformity (§9.5), not in the profile dynamics (§9.6, there being none), and not in the moment map (§9.9). Whether $\mathcal{A} = (0, \frac{1}{3})$ turns on the near-solution alone.

> *Superseded in part.* The near-solution formulation is superseded by §9.16, and the sentence above holds as of this point in the paper. Later sections find exactly one obstruction, and it is to a class rather than to the architecture: the reflection-symmetric midplane cannot be completed inboard (Theorem 9.18). The non-symmetric midplane is then closed (Theorems 9.25–9.26), and what remains open is the matching in height, condition (G).

### 9.11 Constructing a near-solution: what the attempt forces [NUM + EXACT]

§9.10 reduced everything to one near-solution. The attempt is reported here in full, including its failure, because the failure is a theorem.

**The midplane reduction.** Take an $\eta$-independent profile and evaluate on the midplane $\eta = 0$. Then $L = d = 1, H_c = U, W \equiv 1$, and every $\eta$-derivative drops, leaving

$$S_q = -(l + h), \qquad S_n = -\left(D_X U + AU\right),$$

so that, integrating by parts with $U$ compactly supported,

$$c_N = \left(\frac{1}{2} - h\right) \int_0^\infty U \, dX, \qquad c_Q = \int_0^\infty H \, S_q \, dX.$$

**Both moments are single scalars**, linear in $U$ and in $H$ respectively. They are therefore exactly solvable, and in the family $H = \kappa X(1 + (X/X_0)^{\beta})^{-(1+h)/\beta}$ — axis-regular, with the exterior power as its tail — together with a two-bump $U$ of zero integral, **[NUM]** gives $\mu = 1.4 \times 10^{-16}$: machine zero.

**Two structural requirements emerged from failed attempts, and both are exact.**

- **The annulus must lie beyond the angular-momentum maximum.** A first attempt placed it at $a \in [0.07, 0.47]$, where $M_0 = a(a-2) + b_s^2 < 0$ at every point where $b_s = 0$, and $b_s$ must vanish at any interior extremum of $U$. This is §9.4's condition, ignored and then rediscovered numerically. Beyond the maximum $a \to 2 + \alpha$ and $M_0 \geq \alpha(2+\alpha) > 0$ automatically.
- **The circulation must undershoot its ambient power.** For a monotone interpolation $l + h > 0$ everywhere, so $c_Q = -\int H(l+h)\,dX < 0$ **always**: no member of such a family can satisfy the angular moment condition. Vanishing $c_Q$ requires $l < -h$ somewhere — the circulation must fall off *faster* than ambient across the layer and relax back to it. *(This bullet is conditioned on $W \equiv 1$, which holds in the present family because the axial flux is switched off. With the flux restored the condition becomes that $Wl + h$ change sign, and the undershoot relocates strictly inboard of the layer: see §9.16.7(i).)*

With both corrections, $M_0 > 0$ and $M_2 > 0$ on the layer and $\mu$ is machine zero. $M_1$ **fails, and it fails exactly.**

> **Theorem 9.7 (the midplane family is obstructed) [EXACT].** Beyond the support of $U$, $A_X(U) = c_U(\eta)/X$ with $c_U(\eta) := \int_0^{\infty} U\,dX$, so $W = 1 - K/X$ with
>
> $$K(\eta) := 2D\eta\, c_U + d\, c_U'.$$
>
> In the exterior this gives $S_q = -hK/X$ and hence $p_{s,1} = \dfrac{c_Q}{LH} + \dfrac{K}{L}$. For an $\eta$-independent profile at $\eta = 0$, $K \equiv 0$; then $c_Q \neq 0$ sends $p_{s,1} \to \pm\infty$, and $c_Q = 0$ sends $p_{s,1} \to 0$. Neither equals $a$. So the exterior balance $p_s = s$ fails, $w \to -s$, and
>
> $$M_1 = w \cdot s \;\to\; -|s|^2 < 0:$$
>
> **the demanded stress lands in the antipodal half-plane.** Confirmed numerically: $p_{s,1} = 2.5 \times 10^{-4}$ against $a = 2.0200$ at $X = 2000$, and $\min m_1 = -1.0000$ for every axial amplitude tried.

> **Corollary 9.8 (what the axial structure must supply) [EXACT].** Imposing both $c_Q = 0$ and the exterior balance $p_{s,1} = a$ forces $K = La$, i.e.
>
> $$2D\eta\, c_U(\eta) + \left(1 - 2D\eta^2\right) c_U'(\eta) = \left(1 - 2h\eta^2\right)\left(2 + \alpha\right),$$
>
> a first-order linear ODE for the **total axial flux** $c_U$. At the midplane it reads
>
> $$\boxed{c_U'(0) = 2 + \alpha}$$
>
> and together with the axial moment condition $c_U(0) = 0$ this gives $c_U(\eta) = (2+\alpha)\eta + O(\eta^2)$.

**The radially integrated axial flux must vary with height, at a rate fixed exactly by the exterior circulation exponent.** This is the missing state of §9.6 made explicit and quantitative: the axial derivative that no radial state records is not a technical nuisance but carries a specific value that the architecture requires. It also explains structurally why the axial outflow cannot be a passive bookkeeping device — its **height dependence** is load-bearing, and $\alpha$ sets its rate.

> **Status.** The construction is **[NUM]**; Theorem 9.7 and Corollary 9.8 are **[EXACT]** given the transcription of the source terms (§9.9's caveat applies unchanged). The attempt did not produce a near-solution. It produced three requirements on any family that could: the layer beyond the angular-momentum maximum, a circulation undershoot, and a height-varying axial flux with $c_U'(0) = 2 + \alpha$.

## 9.12 What now remains

| Construct $(E, U)$ with $c_U(\eta) = (2+\alpha)\eta + O(\eta^2)$, $l < -h$ on the layer, the layer beyond the maximum, and $M_0, M_1, M_2 > 0$. |
|---|

This is strictly more specific than §9.10's statement and strictly less than Lemma H″: the family is now prescribed in three respects, and only the existence of one member is at issue. Nothing found here obstructs it — Theorem 9.7 obstructs a *family*, not the problem, and names precisely the ingredient that family lacked.

## 9.13 The minimal two-dimensional profile, and what it reformulates [EXACT + NUM]

**The required axial flux has a closed form.** Corollary 9.8's ODE, with $d = L - 2D\eta^2 = 1 - \eta^2$ and $2D = 1 - \alpha$, reads $(1-\eta^2)c_U' + (1-\alpha)\eta\, c_U = (1-\alpha\eta^2)(2+\alpha)$, and is solved exactly by

$$\boxed{c_U(\eta) = (2+\alpha)\,\eta}$$

(residual identically zero, verified symbolically). It is odd, has $c_U(0) = 0$ as the axial moment requires and $c_U'(0) = 2 + \alpha$ as Corollary 9.8 requires, and gives $K = 2D\eta c_U + d c_U' = (2+\alpha)(1-\alpha\eta^2) = La$ **identically**, so the exterior balance closes for every $\eta$. **The radially integrated axial flux is linear in height, with slope** $2 + \alpha$**.**

The minimal profile realizing it is $E = E_0(X)$ with $U(X, \eta) = (2+\alpha)\eta\, u(X)/\bar{u}$: one scalar function of $\eta$, entering linearly. Building it removes Theorem 9.7's obstruction — and exposes two more requirements.

**(i) The odd-flux profile degenerates on the midplane [EXACT].** At $\eta = 0$ it has $U = 0$, hence $b_s = 0$, $H_c = 0$ and $S_n \equiv 0$, so $w_2 = 0$. The state is purely radial: $s = (a, 0)$, $w = (w_1, 0)$, $M_2 > 0$ automatically, and everything reduces to $a > 2$ and $w_1 > 0$. Writing $g := \int_0^X HS_q\, dx - aH$, so that $w_1 = g/H$, one has $g(0) = g(\infty) = 0$ and hence $\int_0^\infty g'\, dX = 0$; and $g'(0^+) < 0$ because $l \to 1$ at the axis. **If** $g'$ **changes sign only once,** $g \le 0$ **throughout and** $w_1 \le 0$ **everywhere.** So $M_1 > 0$ on the midplane requires $g'$ to change sign at least three times — a genuinely non-monotone circulation slope.

Making $l$ non-monotone is not enough, and the failure is robust rather than marginal. **[NUM]** A scan over 48 geometries — undershoot located at $X_d \in \{6, 12, 20, 35\}$ with width $\in \{3, 8\}$, axial bump at $X_u \in \{10, 20, 40\}$ with width $\in \{5, 10\}$, the undershoot depth tuned in each case so that $c_Q = 0$ — gives $w_1 < 0$ at **every** point where $a > 2$, in every case, with

$$\max_{\{a>2\}} w_1 \;\approx\; -1.9 \quad \text{throughout the scan,}$$

against $a \approx 2$: that is $p_{s,1} \approx 0.1$, i.e. the integrated inviscid stress stays near zero while $s$ does not. **The demanded stress sits near** $-s$**, not near the cone.** Shape-tuning inside the class of arbitrary admissible profiles does not approach the boundary, let alone cross it, which is the numerical signature of (ii) below: the profile is far from zero-stress, and no amount of adjusting the circulation undershoot changes that.

This is the degeneracy the primary construction avoids by giving the axial profile a small bias, $v := U(\cdot, 0) \neq 0$, for the reason it states: reflection symmetry would leave no radial shear of the axial velocity at the midplane. *An earlier version of this analysis read this computation as proving* $v \neq 0$ *necessary; that inference is withdrawn.* The scan is evidence about the family scanned, and §§9.16.3–9.16.5 construct an active layer with $v \equiv 0$. The correctly scoped necessity is **Theorem 9.18**, with its own proof: $v \not\equiv 0$ is required *inboard* of the layer for a globally admissible midplane, while $v = 0$ is allowed on the layer itself.

**(ii) Adding the bias is not sufficient [NUM] (necessity: see Theorem 9.18, not this paragraph).** Writing $U = (2+\alpha)\eta\, u/\bar{u} + v(X)$ with $v$ even and $\int v\, dX = 0$ — so $c_U$ is unchanged — restores $b_s \neq 0$ and $w_2 \neq 0$ at $\eta = 0$. But $\min m_1$ stays at $-1$ for bias amplitudes from 0.5 to 8, with $|b_s|$ reaching 464. The reason is structural:

$$M_1 = w \cdot s = w_1 a - w_2 b_s, \qquad w_2 = p_{s,2} + b_s,$$

so a large $b_s$ **hurts** unless $p_{s,2}$ tracks $-b_s$ — that is, unless $w_2$ stays small. And $p_{s,2} = -b_s$ is precisely the axial half of the zero-stress relation. *w* **is not an arbitrary residual; it is the departure from a zero-stress profile, and it must stay small and correctly directed.**

**The reformulation this forces.** Arbitrary profiles will not do: one must start from a solution $\Phi_*$ of the vanishing-residual equations, where $w = 0$ identically, and perturb,

$$\Phi = \Phi_* + \epsilon\Phi_1 \quad \Rightarrow \quad w = \epsilon\,\mathcal{L}[\Phi_1] + O(\epsilon^2),$$

with $\mathcal{L}$ **linear**. Since $\mathcal{C}(s)$ is an open convex cone (Theorem 9.1) and $s$ is fixed at order $\epsilon^0$ by $\Phi_*$, cone membership becomes a condition on the *direction* of $\mathcal{L}[\Phi_1]$ alone:

| find $\Phi_1$ supported in the layer, preserving the moments, with $\mathcal{L}[\Phi_1](X) \in \mathcal{C}\left(s_*(X)\right)$ for all $X \in I$. |
|---|

**A linear feasibility problem in a convex cone**, at fixed background — not the nonlinear search this section attempted three times. That is the correct formulation of the remaining step, and it is the first time the problem has been reduced to a linear one.

> **Status.** $c_U = (2+\alpha)\eta$ and the midplane degeneracy are **[EXACT]**; the bias computation is **[NUM]**; the linear reformulation is **[EXACT]** given differentiability of the residual map. The transcription caveat of §9.9 applies throughout. **No near-solution was constructed.** What was produced is a fourth requirement — proximity to a zero-stress profile — and a reformulation that makes the target linear. The 48-geometry scan is the evidence that the fourth requirement is not optional: within arbitrary admissible profiles the margin is not merely negative but uniformly far from zero, and the distance is explained exactly by the departure from zero-stress.

### 9.14 The succession, recorded

Three simplifications have now been tried and have failed, and each failure named the dimension that had been flattened out of the problem:

| | | |
|---|---|---|
| stress-driven core | $\xrightarrow{\text{fail}}$ | stress belongs to the matching layer, |
| finite radial trajectory | $\xrightarrow{\text{fail}}$ | full axial profile state required, |
| $\eta$-independent near-solution | $\xrightarrow{\text{fail}}$ | axial variation structurally required, |
| arbitrary admissible profile | $\xrightarrow{\text{fail}}$ | proximity to zero-stress required. |

The pattern is uniform and is the operational content of the rules of §1.3: **a simplification that removes a distinction the system is using does not produce a wrong answer, it produces an obstruction that names the distinction.** Each row above was found by carrying a reduction to a contradiction rather than by inspecting it, which is why §11.1 records the rules as diagnostic rather than predictive. The target that survives is correspondingly precise:

| Construct $(E,U)(X,\eta)$: $\quad l < -h$ on the layer, $\quad$ layer beyond the $\Gamma$-maximum, $\quad c_U = (2+\alpha)\eta$, |
|---|

| $u_z(r,0) \neq 0$, $\quad \Phi$ near a zero-stress solution, $\quad M_0, M_1, M_2 > 0$, $\quad \lvert\mathcal{M}\rvert \ll \delta_0 W^2$. |
|---|

That target is superseded in §9.15, where the last of the four restorations — the axial-flux profile as a resolution independent of the midplane bias — makes the problem linear and feasible, and the five features collapse to one. Two entries of the box are also corrected later. The requirement $l < -h$ *on the layer* was an artifact of switching off the axial flux (§9.16.7(i)). And $u_z(r,0) \neq 0$ is not required on the active layer (§§9.16.3–9.16.5), but is required inboard of it, by a different proof (Theorem 9.18).

## 9.15 The tangent problem is solvable, and the cone stops being the obstruction

§9.13 reduced the remaining step to a linear question at a fixed background: find $\Phi_1$ supported in the layer, preserving the moments, with $\mathcal{L}[\Phi_1](X) \in \mathcal{C}(s_*(X))$ for all $X \in I$. That question is settled here, affirmatively.

**The linearisation is explicit [EXACT].** On the midplane the profile state is the triple $\Phi = (E, v, U_\eta)$ with $v := U(\cdot, 0)$ and $U_\eta := \partial_\eta U(\cdot, 0)$ — three functions, by §9.6 — and

$$S_q = -Wl - h, \qquad S_n = -WD_X v - Av - vU_\eta, \qquad W = 1 - A_X(U_\eta).$$

Writing $\psi := \delta E/E$, the first variation is

$$\delta S_q = -W\, D_X\psi + l\, A_X(\delta U_\eta), \qquad \delta S_n = -WD_X\delta v - (A + U_\eta)\delta v - v\,\delta U_\eta + A_X(\delta U_\eta)\, D_X v,$$

and $\delta w$ satisfies the driven system of §9.6, $D_X\delta w_1 + l\,\delta w_1 = \delta\Sigma_q$ and $D_X(E\,\delta w_2) = \delta\Sigma_n$, whose integrating factors are $H$ and $E$ respectively. Validated against a centred difference of the nonlinear residual map: relative error $1.6 \times 10^{-8}$ at $\epsilon = 10^{-4}$, and the assembled matrix reproduces $\mathcal{L}$ to $2.8 \times 10^{-13}$.

> **Theorem 9.9 (tangent cone-feasibility).** Let $\Phi_*$ be admissible with $M_0 > 0$ on a compact layer $I$ (for which $a_* > 2$ suffices). Then there is a perturbation $\Phi_1 = (\psi, \delta v, \delta U_\eta)$ supported in the interior of $I$ such that **(i)** $\delta w := \mathcal{L}[\Phi_1](X) \in \mathcal{C}(s_*(X))$ for every $X \in I$, with an angular margin bounded below uniformly on $I$; and **(ii)** $\delta c_Q = \delta c_N = 0$ — both moments are preserved.

**[NUM]** Verified in 64 **of** 64 admissible backgrounds: $\alpha \in \{0.02, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.32\}$, four layer positions $I \subset [8, 55]$, two circulation-dip amplitudes. In every case $\delta w$ lies strictly inside $\mathcal{C}(s_*)$ at **every** grid point of $I$ — up to 10 003 points, with no exclusions — with

$$\theta_c - \angle\big(\delta w, s_*\big) \;\geq\; 0.18 \text{ rad uniformly on } I,$$

moments preserved to $10^{-10}$–$10^{-13}$, and $\Phi_1$ identically zero outside $I$.

**Why it is decidable rather than searched.** Once $s_*$ is fixed at order $\epsilon^0$, membership $T \in \mathcal{C}(s)$ is equivalent to the two **linear** inequalities $T \cdot s > 0$ and $|T \cdot s^\perp| < \tan\theta_c\,(T \cdot s)$, and the moment conditions are linear. Feasibility is therefore a linear program over the coefficients of $\Phi_1$, and a linear program does not have to be searched: it is solved, or a certificate of infeasibility is produced. Three nonlinear searches in §§9.11–9.13 failed where this succeeds, and the difference is not effort but formulation.

> **Corollary 9.10.** If in addition $w_* = 0$ on $I$, then $\Phi_* + \epsilon\Phi_1$ satisfies $M_0, M_1, M_2 > 0$ pointwise on $I$ for all sufficiently small $\epsilon > 0$, because $\mathcal{C}$ is open (Theorem 9.1) and the margin is uniform.

**[NUM]** The cone membership is not merely infinitesimal: the **full nonlinear** increment $\epsilon^{-1}(w(\Phi_* + \epsilon\Phi_1) - w(\Phi_*))$ stays strictly inside $\mathcal{C}(s_*)$ at every point of $I$ for $\epsilon$ up to $10^{-1}$, with the angular margin degrading only from 0.319 to 0.295 rad. Below $\epsilon \approx 10^{-2}$ the finite-difference test loses resolution at the few points where $|\delta w| \sim 10^{-6}$ — cancellation, not geometry; there the validated linearisation is the reliable statement.

> **Corollary 9.11 (conditional resolution of Lemma $H''$).** Suppose there exists an admissible profile $\Phi_*$ — axis regularity, exterior matching, the moment conditions — whose residual vanishes on a compact layer $I$ carrying $M_0 > 0$. Then Lemma $H''$ holds: $\Phi_* + \epsilon\Phi_1$ has $M_0, M_1, M_2 > 0$ on $I$ for small $\epsilon > 0$ and satisfies the same moment conditions, $\Phi_1$ preserving them.

| The cone is reduced to a linear cone-intersection theorem. What remains: prove that, and exhibit a zero-stress base with $\alpha_{\text{loc}} > 0$. |
|---|

**Both are supplied in §9.16**: the base is constructed, and at it the linear theorem has a closed-form solution (Theorem 9.13) rather than a numerical one. The hierarchy below is the status of the *general* tangent statement, which remains numerical; the status at the constructed base is stronger.

**The hierarchy, stated exactly.**

| | |
|---|---|
| **[EXACT]** | the linearisation, and the reformulation of cone membership as a linear programme |
| **[NUM]** | that programme is feasible on 64/64 tested backgrounds |
| **[COND]** | *if* tangent feasibility holds at a zero-stress base, then nonlinear realizability follows after the $O(\epsilon^2)$ moment repair of Theorem 9.5 |
| **[EXACT]** | at the base of §9.16, tangent feasibility is closed-form (Theorem 9.13) |
| **[OPEN]** | tangent feasibility analytically at a *general* background; and $\eta \neq 0$ |

That is one object with one scalar condition, in place of a family prescribed in five respects (§9.14). It is also exactly the fourth requirement §9.13 extracted from the failed searches, now shown to be not merely necessary but **sufficient**, given Theorem 9.9.

**The margin that matters is angular, and two earlier runs were corrupted by ignoring that.** $|\delta w|$ varies by six orders of magnitude across $I$ and vanishes at both edges, so $M_1$ and $M_2$ are small in absolute terms ( $\min M_1 \approx 8 \times 10^{-7}$, $\min M_2 \approx 2 \times 10^{-12}$ at the reported normalisation) while the scale-invariant margin is uniformly $O(10^{-1})$. A coarse constraint grid certifies nothing between its samples, and an **absolute** feasibility tolerance certifies nothing where the row coefficients are below it: both produced spurious verdicts here, in opposite directions, before the constraints were row-normalised and the verification moved onto every grid point. The cone condition is homogeneous in $\delta w$, which is what makes the normalisation legitimate.

**(a) The axial-flux profile must move, and must move independently [NUM].** With $\delta U_\eta \equiv 0$ the programme is **infeasible** in all 12 backgrounds tested; with $\delta U_\eta$ tied to $\delta v$ it is **infeasible**; and $\delta U_\eta \equiv 0$ remains infeasible **even with both moment constraints dropped**. So what forbids the reduced parameterisations is the cone itself, not the moment bookkeeping. The midplane bias $v$ and the axial-flux profile $U_\eta$ are independent resolutions of the state, and identifying them — which each earlier attempt in this section did implicitly — is what made the problem appear rigid. This is §9.6's structure theorem appearing a second time, now as a condition on the admissible perturbation rather than on the state.

**(b) In the tested non-zero-stress family, the residual cannot be placed on the cone axis [NUM, that family only].** Imposing $|\delta w \cdot s^{\perp}| \leq \kappa\,(\delta w \cdot s)$ is feasible for $\kappa \geq 0.20\tan\theta_c$ and infeasible for $\kappa \leq 0.15\tan\theta_c$. The achievable residual directions form a **proper** subcone of $\mathcal{C}(s_*)$: the misalignment is bounded below. There is therefore no optimal tangent perturbation *in that family*, and there the covariance constraint is **satisfiable but not saturable**. This is not a statement about the architecture: at the zero-stress layer base of §9.16, Theorem 9.13 places the residual exactly on the axis, $w = \varepsilon(\chi, 0)$. An early calculation here claimed $\delta w$ could be put exactly on the axis with $J_c = 0$; that was an artifact of imposing neither the moments nor the requirement that $\delta w$ come from an actual perturbation, and it is withdrawn.

**(c) What this is not.** The backgrounds $\Phi_*$ used in the tests are **not** zero-stress: $|w_*| \in [1.15, 2.58]$ on $I$ and $w_* \cdot s_* < 0$ throughout. Theorem 9.9 is a statement about $\delta w$, not about $w_* + \epsilon\,\delta w$, and Corollary 9.11 is conditional for exactly that reason. No near-solution is exhibited.

**The succession, continued.** The fifth row differs in kind from the four of §9.14:

$$\text{linear tangent problem} \xrightarrow{\textbf{solvable}} \text{the cone is not the obstruction.}$$

Each of the four flattenings produced an obstruction that named the distinction it had removed. Restoring the last of them — $\delta U_\eta$ as a resolution independent of $\delta v$ — converted the problem from a search into a decidable linear programme, and it was decided. That is the method's claim in its positive form, and it is the first point in this investigation at which the sequence terminated in a construction rather than in a further requirement.

> **Status.** The linearisation is **[EXACT]** and validated; Theorem 9.9, the finite-$\epsilon$ range and the facts (a)–(b) are **[NUM]**; Corollaries 9.10–9.11 are **[EXACT]** given Theorem 9.9 and the openness of $\mathcal{C}$. The transcription caveat of §9.9 applies throughout. **Nothing here is an existence proof for Lemma $H''$.** What it does is remove the cone from the list of obstacles and leave exactly one.

## 9.16 The midplane, closed analytically

§9.15 left one object to find. Writing down what "zero stress" *means* produces it, together with a structural obstruction that explains the shape of everything before it; the numerical shooting condition that first appeared here has since been eliminated (Theorem 9.14), and Theorem 9.13 turns out to be a finite rather than a tangent statement. Everything in this section is **[EXACT | TRANSCRIPTION]** — proved algebraically from the transcribed source equations, and so inheriting the caveat of §9.9, but not resting on any numerical step.

### 9.16.1 The layer equation, and what a stress-free region cannot do

On the midplane $w_1 \equiv 0$ is equivalent to $\Sigma_q \equiv 0$, and with $a = 2 - 2l$ and $S_q = -Wl - h$ that is a first-order ODE for $l$ alone:

$$2X\,l' \;=\; 2l(1-l) + XWl + Xh.$$

> **Lemma 9.12 (no downward crossing).** At $l = 0$, (Z) gives $l' = h/2 > 0$ — **independently of $W$, of $v$, and of every other field.** Since $H \sim X$ at the axis gives $l(0^+) = 1 > 0$, a profile that is zero-stress on $(0, X]$ has $l > 0$ there, hence $\alpha_{\mathrm{loc}} = -2l < 0$ throughout.

**A globally zero-stress profile therefore carries $\alpha_{\mathrm{loc}} > 0$ nowhere.** The stress-free region cannot *make* a positive exterior exponent; it can only inherit one across its boundary. That is the supply theorem of §6 again, one level down and now for the exponent rather than the swirl: the interior inherits through its boundary what it cannot generate. Note also that $l = 0$ becomes a fixed point exactly at $h = 0$ — the layer equation degenerates at $\alpha \to 0$, where §9.4 found the cone degenerating. Same parameter, third appearance.

The structural consequence is that **the preparation region and the layer must be kept distinct**: the inboard region is *required* to carry nonzero $\Sigma_q$, because its job is to manufacture boundary data the layer cannot generate internally.

$$\text{axis-regular inboard preparation} \;\to\; l = -h \text{ at } X_a \;\to\; \text{zero-stress stationary layer.}$$

### 9.16.2 The stationary layer, and the flux constant

Rewriting (Z) in the exponent,

$$D_X\alpha_{\mathrm{loc}} \;=\; \alpha_{\mathrm{loc}}\Big(1 + \tfrac{\alpha_{\mathrm{loc}}}{2}\Big) \;+\; \tfrac{X}{2}\big(W\alpha_{\mathrm{loc}} - \alpha\big),$$

and asking for $\alpha_{\mathrm{loc}} \equiv \alpha$ constant — which is what matching a power-law exterior means — forces $1 - W = (2+\alpha)/X$, that is

$$\boxed{\int_0^X U_\eta\, dx \;=\; 2 + \alpha \quad \text{on the layer.}}$$

**Corollary 9.8 gives $c_U'(0) = \int U_\eta\, dX = 2 + \alpha$, derived independently from the exterior axial balance.** The same constant, reached from the radial layer equation. The axial flux is therefore carried entirely *inboard* of the layer, and its total is fixed. One verifies directly that $l \equiv -h$ with $W = 1 - (2+\alpha)/X$ solves (Z) with residual identically zero.

**The base.** On $I = [X_a, X_b]$ take $l \equiv -h$ (so $a \equiv 2 + \alpha$ and $M_0 = \alpha(2+\alpha) > 0$), $v \equiv 0$, and $U_\eta \equiv 0$, with total flux $2 + \alpha$ deposited inboard. Since every term of $S_n = -WD_X v - Av - vU_\eta$ carries a factor $v$ or $D_X v$, we get $S_n \equiv 0$, hence $b_s \equiv 0$, $w_2 \equiv 0$ and $c_N = 0$ — **exactly, with no quadrature.** One scalar condition remains, $w_1(X_a) = 0$, and §9.16.4 disposes of it.

### 9.16.3 Theorem 9.13 is finite, not tangent

> **Theorem 9.13 (finite cone-axis perturbation).** On the base above let $\chi$ be smooth with $\mathrm{supp}\chi \subseteq [X_a, X_b]$ and set
>
> $$\psi := 0, \qquad \delta v := 0, \qquad \delta U_\eta := -\tfrac{1}{h}\big(D_X\chi - h\chi\big)'.$$
>
> Then for **every** $\varepsilon \in \mathbb{R}$, with $\Phi_\varepsilon := \Phi_* + \varepsilon\Phi_1$,
>
> $$w(\Phi_\varepsilon) = \varepsilon\,(\chi, 0) \quad \text{on } I, \qquad c_Q(\Phi_\varepsilon) = c_Q(\Phi_*), \qquad c_N(\Phi_\varepsilon) = c_N(\Phi_*) = 0,$$
>
> and $E, H, l, a, b_s$ are unchanged. There is no $O(\varepsilon^2)$ remainder and no smallness condition.

*Proof.* $\psi = 0$ leaves $E, H, l, a$ untouched and $\delta v = 0$ leaves $v \equiv 0$, hence $b_s \equiv 0$. Because $A_X$ is linear and $U_\eta$ is affine in $\varepsilon$, $W(\Phi_\varepsilon) = W_* - \varepsilon A_X(\delta U_\eta)$ is **affine**; and $S_q = -Wl - h$ is **linear in** $W$ at fixed $l$, so

$$S_q(\Phi_\varepsilon) = S_{q,*} + \varepsilon\, l\, A_X(\delta U_\eta), \qquad \partial_\varepsilon^2 S_q \equiv 0.$$

Every term of $S_n$ carries $v$ or $D_X v$, both identically zero for all $\varepsilon$, so $S_n(\Phi_\varepsilon) \equiv 0$ and $w_2 \equiv 0$. Since $w_1 = H^{-1}\int_0^X HS_q - a$ is linear in $S_q$ at fixed $H, a$, $w_1$ is affine in $\varepsilon$; $A_X(\delta U_\eta) = -(D_X\chi - h\chi)/(hX)$ gives $\delta\Sigma_q = X\,\delta S_q = D_X\chi - h\chi$, and with $l = -h$ the function $\chi$ is the unique solution of $D_X\delta w_1 + l\,\delta w_1 = \delta\Sigma_q$ vanishing at $X_a$. Finally $H \propto X^{-h}$ gives $H' = -hH/X$, so

$$\delta c_Q = \int H\,\delta S_q\, dX = \int \Big(H\chi' - \tfrac{hH}{X}\chi\Big)\, dX = \int (H\chi)'\, dX = \big[H\chi\big] = 0,$$

and $\delta c_N = \int \delta S_n\, dX = 0$. ▫

Because $a$ and $b_s$ are unchanged, $s_* = (2+\alpha, 0)$ is the **same** cone for every $\varepsilon$, and $w = \varepsilon\chi\, s_*/(2+\alpha)$ lies on its axis: $J_c = 0$ exactly, with angular margin the full half-angle $\theta_c = \arctan\sqrt{2/\alpha}$. Numerically the identity holds to $10^{-9}$ relative at $\varepsilon = 10^3$ — a check on the algebra, not its basis.

**The edges vanish because they are forced to, not because $\chi$ was chosen compactly supported.** Outboard of $\mathrm{supp}\chi$ the source vanishes, so $D_X w_1 + l w_1 = 0$ and $w_1 H$ is constant; since $aH \to 0$, $c_Q = \lim(w_1 + a)H = w_1 H\big|_{\text{out}}$. The moment condition $c_Q = 0$ therefore forces $w_1 \equiv 0$ outboard, and continuity of $w_1$ gives $\chi(X_b) = 0$; the base preparation gives $\chi(X_a) = 0$. Hence:

> **The demanded stress necessarily vanishes at both edges of the layer.** The cone conditions are conditions on a *nonzero* demand, so the correct statement is strict positivity on the **open active layer** $I_\chi := \{\chi > 0\}$, with $M_1 = M_2 = 0$ at $\partial I_\chi$ where the demand itself is zero. $M_0 = \alpha(2+\alpha) > 0$ holds on all of $I$ regardless of $\chi$.

This corrects the wording of an earlier version, which claimed strict positivity "on the whole layer" while using a compactly supported $\chi$ — not literally compatible. It is a strengthening rather than a retreat: the edge vanishing is now a theorem about the moment condition instead of an artifact of the test function.

### 9.16.4 The Base Preparation Lemma, proved

**The control functional.** Using $H' = Hl/X$ and integrating $\int WXH'$ by parts with $\Psi := \int_0^X U_\eta$, $\Psi(X_a) = 2 + \alpha$:

$$\boxed{w_1(X_a)\,H(X_a) \;=\; -X_a H(X_a) \;+\; (1-h)\int_0^{X_a} H\,dX \;-\; \int_0^{X_a} U_\eta H\,dX}$$

so the base condition is exactly $(1-h)\int_0^{X_a} H\,dX - \int_0^{X_a} U_\eta H\,dX = X_a H(X_a)$. **[EXACT]**; verified against direct evaluation with the residual scaling as $O(\Delta X)$, an endpoint quadrature artifact.

**The design.** Fix $0 < X_t < X_\beta < X_a$. Let $l_0$ be smooth and axis-regular with $l_0(0^+) = 1$, $l_0 \le 1$, and $l_0 \equiv -h$ on $[X_t, \infty)$. Let $\beta \ge 0$ be smooth, $\beta \not\equiv 0$, supported in $(X_t, X_\beta)$; let $U_\eta \ge 0$ be smooth, supported in $(X_\beta, X_a)$, with $\int U_\eta = 2 + \alpha$. Put $l_\lambda := l_0 - \lambda\beta$, so $H_\lambda = H_0 e^{-\lambda B}$ with $B := \int_0^X \beta\,dx/x$ nondecreasing and $B \equiv B_\infty$ beyond supp$\beta$. **The two supports are disjoint, and that is what makes the next line clean.**

> **Theorem 9.14 (Base Preparation).** With $R_\lambda := H_\lambda/H_\lambda(X_a)$ and $F(\lambda) := w_1(X_a)[l_\lambda]$,
>
> $$F'(\lambda) = (1-h)\int_0^{X_a} \big(B_\infty - B(X)\big) R_\lambda(X)\,dX \;>\; 0,$$
>
> $F(\lambda) \to +\infty$ as $\lambda \to +\infty$, and
>
> $$F(\lambda) \;\to\; -X_a^h X_\beta^{1-h} - C \;<\; 0 \quad (\lambda \to -\infty), \qquad C := \int U_\eta \frac{H_0}{H_0(X_a)} dX > 0.$$
>
> Hence there is a **unique** $\lambda_* \in \mathbb{R}$ with $F(\lambda_*) = 0$.

*Proof.* $R_\lambda = (H_0/H_0(X_a))e^{\lambda(B_\infty - B)}$, so $\partial_\lambda R_\lambda = (B_\infty - B)R_\lambda \ge 0$, strictly positive inboard of supp$\beta$. Differentiating $F = (1-h)\int R_\lambda - \int U_\eta R_\lambda - X_a$, the second integral contributes $\int U_\eta (B_\infty - B) R_\lambda$, which **vanishes identically** because $B \equiv B_\infty$ on supp$U_\eta$ — this is exactly why $U_\eta$ is placed outboard of $\beta$. That gives $F' > 0$. For $\lambda \to +\infty$, $R_\lambda \to \infty$ on $\{B < B_\infty\}$, a set of positive measure. For $\lambda \to -\infty$, $e^{\lambda(B_\infty - B)} \le 1$ dominates and converges to $\mathbf{1}_{\{B = B_\infty\}}$, so by dominated convergence $(1-h)\int R_\lambda \to (1-h)\int_{X_\beta}^{X_a} (X/X_a)^{-h} dX = X_a - X_a^h X_\beta^{1-h}$, using $l_0 \equiv -h$ there; the $U_\eta$ term is $-C$ for every $\lambda$. Subtracting $X_a$ gives the stated limit, which is strictly negative. Monotonicity and the intermediate value theorem give existence and uniqueness. ▫

**The numerical shooting of an earlier version is therefore withdrawn and replaced.** It was never more than a one-dimensional analytic problem, and $F$ is not merely sign-changing but strictly monotone, so the preparation parameter $\lambda_*$ is *unique within the chosen one-parameter family* $l_\lambda = l_0 - \lambda\beta$ at fixed $(l_0, \beta, U_\eta, X_a)$, not merely existent. Uniqueness of the base across admissible choices of $(l_0, \beta, U_\eta)$ is not claimed: each choice carries its own $\lambda_*$. For $\alpha = 0.2$, $X_a = 20$, $X_\beta = 11$ the root is $\lambda_* = 1.9008\ldots > 0$: a genuine circulation **undershoot**, sitting strictly inboard of the layer.

### 9.16.5 The midplane theorem

> **Theorem 9.15 (Midplane Realizability).** For every $\alpha \in (0, \frac{1}{3})$ and every admissible layer location there is a smooth, axis-regular midplane profile $(E, v, U_\eta)$ satisfying the moment conditions $c_Q = c_N = 0$, with zero demanded stress at the layer edges and
>
> $$M_0,\, M_1,\, M_2 \;>\; 0 \qquad \text{throughout the open active layer } I_\chi.$$

*Proof.* Theorem 9.14 supplies the base with $w_1(X_a) = 0$; §9.16.2 makes the layer zero-stress with $a \equiv 2 + \alpha$, so $M_0 = \alpha(2+\alpha) > 0$; Theorem 9.13 adds a finite perturbation giving $w = \varepsilon\chi\, s_*/(2+\alpha)$ with both moments preserved exactly, so $M_1 = \varepsilon\chi|s_*|^2/(2+\alpha) > 0$ and $J_c = 0$, whence $M_2 = 2aM_1^2 > 0$, on $\{\chi > 0\}$. ▫

**Lemma $H''$ is solved analytically on the active midplane layer — not on the whole midplane fibre** (**Theorem 9.18**).

**Status: [EXACT | TRANSCRIPTION].** Every step is algebraic. What it is *not* is source-independent: the transcription caveat of §9.9 governs the whole chain, and if the source equations are transcribed wrongly the theorem is about the wrong system. Nothing here is **[NUM]** any longer; the numerics that remain in this section are confirmations of algebra, and are labelled as such.

### 9.16.6 Obstruction test

Run against the construction, seven candidate obstructions, all absent, each for a stated reason.

| **candidate** | **status** |
|---|---|
| inboard profile loses axis regularity | absent: $\beta$ and $U_\eta$ are supported away from the axis, so $l(0^+) = 1$ is untouched |
| $E$ changes sign | impossible: $E = H/\sqrt{2X}$ with $H = \exp\int l\,dx/x > 0$ for any real $l$ |
| axial flux cannot be both $2+\alpha$ and smooth | absent: $U_\eta$ is any smooth bump on $(X_\beta, X_a)$ with that integral |
| $w_1(X_a) = 0$ unreachable without breaking the join | absent: $\mathrm{supp}\beta \subset (X_t, X_\beta)$ with $X_\beta < X_a$, so $l \equiv -h$ on both sides of $X_a$ for every $\lambda$ |
| the finite perturbation destroys the stationary relation | **true and intended**: $\Psi \neq 2+\alpha$ on $I$ after perturbing, which *is* the stress. But $l$, hence $a$, hence $s_*$ and $M_0$, are untouched |
| the moment identities cease to hold exactly | absent: both are exact for all $\varepsilon$ (Theorem 9.13) |
| strict cone membership fails near the edges | **true, and forced** — see §9.16.3; the theorem is stated on $I_\chi$ accordingly |

Two features worth stating rather than hiding. **(a)** At $\lambda_*$ the inboard exponent is large ($a$ up to $\approx 3.6$, i.e. $\alpha_{\mathrm{loc}} \approx 1.6$). That is a **core** feature and no violation of the window, which constrains the exterior exponent at the core edge — here $\alpha_{\mathrm{loc}} = \alpha$ on the layer, by construction. **(b)** For large $\varepsilon$, $U_\eta$ becomes sign-changing on the layer. $U_\eta = \partial_\eta U|_{\eta=0}$ is a velocity *gradient*, not a density, so no positivity is required of it; the conserved quantity $\int U_\eta\,dX = 2+\alpha$ is preserved exactly.

### 9.16.7 Earlier claims this corrects

**(i)** §9.11 concluded that the circulation must undershoot *on the layer*, $l < -h$ there. That was derived with $W \equiv 1$, i.e. with the axial flux switched off, where $c_Q = -\int H(l+h)\,dX$. With the flux restored the condition is that $Wl + h$ change sign, and the undershoot **moves inboard of the layer**; on the layer $l = -h$ exactly. It is the same object — $\lambda_* > 0$ is precisely an undershoot — in a different place.

**(ii)** §9.13 read its midplane degeneracy as proving $v := U(\cdot, 0) \neq 0$ *necessary*. As a proof, that is withdrawn: here $v \equiv 0$ on the active layer in both base and perturbation, and $M_1 > 0$ regardless. The necessity survives in corrected scope and with a different proof — **Theorem 9.18**: $v \not\equiv 0$ is required *inboard* of the layer for a globally admissible midplane. So $v = 0$ is allowed on the active layer, and $v \not\equiv 0$ is required in the global preparation. What §9.13(i) actually proves — that $g' = (w_1 H)'$ must change sign at least three times — is satisfied: the design of §9.16.4 gives **four** sign changes above the numerical noise floor, comfortably inside the bound (an earlier, superseded design gave exactly three; the count is design-dependent and only the lower bound is structural). The necessary condition was right; the sufficient-sounding gloss on it was not.

**(iii)** §9.15(b) reported that the residual cannot be placed on the cone axis. That is a property of the **non-zero-stress** backgrounds tested there, where $b_s \neq 0$; at the zero-stress base $b_s \equiv 0$, the axis is radial, and $\delta w_2 = 0$ puts the residual exactly on it. The statement stands for the tested family and is withdrawn as a general one.

All three have the same shape: a restriction adopted for tractability — no axial flux, a tied bias, a non-zero-stress background — turned into an apparent necessity. That is the failure mode the rules of §1.3 exist to catch, and it caught it only because the reductions were carried to contradiction rather than inspected.

### 9.16.8 Off the midplane: the corner geometry, and the global obstruction it exposed

An earlier version concluded from a parity expansion that the admissible region is a wedge $|\eta| < c\varepsilon\chi(X)$ and that "the layer endpoints must move with height". That conclusion is treated here as a conjecture and tested against the full source system in both directions. **It does not survive.** The wedge is withdrawn, moving edges are not forced, and the test exposed something §§9.16.1–9.16.5 had not checked: that the midplane closure is local to the layer.

**(a) Parity is a class, not a consequence [CLASS].** $U(X,0) = 0$ does not imply that $U$ is odd; the implication runs the other way. Under **[CLASS-PARITY]**, $E(X,-\eta) = E(X,\eta)$ and $U(X,-\eta) = -U(X,\eta)$, the quantities $W, L, d, l, a, S_q, w_1$ are even and $H_c, (\log E)_\eta, S_n, b_s, w_2$ are odd. Under **[GENERAL]** the midplane still fixes more than its values: when $U(\cdot,0) = 0$ the source gives $S_n|_{\eta=0} = -\Pi_\eta|_{\eta=0}$, so the fixed midplane, which has $w_2 \equiv 0$ at $\eta = 0$, forces $\Pi_\eta|_0 \equiv 0$ and hence $\partial_\eta E|_0 \equiv 0$. The one remaining first-order jet, $U^{(2)} := \partial_\eta^2 U|_0$, enters only $\partial_\eta S_q|_0 = l\, A_X(U^{(2)})$ — the coefficient of $\eta$ in $w \cdot s$ — where near an edge it can only do harm. **The first-order corner problem is therefore the same in both classes.** *(Verified symbolically.)*

**(b) The transverse jet [EXACT | TRANSCRIPTION].** Write $U^{(1)} := \partial_\eta U|_0$ and $E_2 := \partial_\eta^2 E|_0$, and normalise the pressure at radial infinity — which exterior matching forces, up to an offset $\kappa_0(1-\eta^2)^{-2A}$ that cancels identically from $S_n$. Then

$$\partial_\eta S_n|_0 = -W\, D_X U^{(1)} - U^{(1)} - \left(U^{(1)}\right)^2 + P_0 - \Pi_2, \qquad P_0 := 4A\Pi_0 + 2D_X\Pi_0, \qquad \Pi_2 = -\int_X^\infty \frac{E_0 E_2}{x}\, dx,$$

$$E_0\, \partial_\eta w_2 = \int_0^X \partial_\eta S_n|_0\, dx + 2X\, U^{(1)'}, \qquad B_{01} := \partial_\eta\left(w \cdot s^\perp\right)|_0 = w_1\, \partial_\eta b_s|_0 + (2+\alpha)\, \partial_\eta w_2.$$

The term $-\Pi_2$ comes from $-d\,\Pi_\eta$ in the source's $S_n$. **An intermediate version of this computation omitted it**, and concluded that no off-midplane degree of freedom can reach $B_{01}$ and that the two corners cannot both be repaired. Both conclusions are withdrawn: the axial curvature of the pressure reaches $B_{01}$ directly. This is the transcription caveat of §9.9 materialising, and it was caught by re-reading the source rather than by any internal check.

**(c) The first-order axial moment forces the pressure to bend [EXACT | TRANSCRIPTION].** Integrating (b) over $(0,\infty)$, one integration by parts in each group gives

$$\partial_\eta c_N|_0 = -2\int_0^\infty \left(U^{(1)}\right)^2 dX \;+\; 2\alpha \int_0^\infty \Pi_0\, dX \;+\; \int_0^\infty E_0 E_2\, dX.$$

The first two terms are strictly negative for every admissible midplane profile, since $\int U^{(1)} = 2 + \alpha$ and $\Pi_0 < 0$. Hence:

> **Corollary 9.16 (the pressure must bend).** A continuation of a midplane profile with $U(\cdot,0) = 0$ satisfies $c_N(\eta) = 0$ to first order only if
>
> $$\int_0^\infty E_0\, \partial_\eta^2 E|_0\, dX \;=\; 2\int_0^\infty \left(U^{(1)}\right)^2 dX - 2\alpha \int_0^\infty \Pi_0\, dX \;>\; 0.$$

A height-independent circulation profile cannot leave the midplane even at first order. **[NUM]** The identity agrees with a centred difference of the nonlinear map $\eta \mapsto c_N(\eta)$ to $1.6 \times 10^{-8}$ relative, with $\partial_\eta E|_0$ and $E_2$ both deliberately nonzero; the same run confirms $S_n|_0 = -\Pi_\eta|_0$ pointwise.

**(d) The natural continuation, and why its truncations cannot decide [EXACT | TRANSCRIPTION + NUM].** Continue Theorem 9.13 with $E_2$ supported inboard of $X_a$, so that $\Pi_2 \equiv 0$ on the layer. There $E_0 \propto X^{-A}$, so $P_0 \equiv 0$, and since $\int_I \delta U_\eta = 0$ the layer integral collapses to a square:

$$\int_I \Big( -WD_X U^{(1)} - U^{(1)} - \big(U^{(1)}\big)^2 \Big) dX = -2\varepsilon^2 \parallel \delta U_\eta \parallel^2_{L^2(I)} .$$

The moment condition then gives $B_{01}(X_b) = 0$ and $E_0\, \partial_\eta w_2(X_a) = 2\varepsilon^2 \parallel \delta U_\eta \parallel^2 > 0$. For $\chi \sim c|\xi|^m$ at the edges, $\xi := X - X_e$, the first-order admissible set near the corners is

$$|\eta| < \kappa_b\, |X_b - X|^3, \quad \kappa_b = \frac{\sqrt{\alpha}\; E_0(X_b)}{2\sqrt{2}\; X_b^2\, m(m-1)(m-2)}; \qquad |\eta| < \kappa_a\, \chi(X), \quad \kappa_a = \frac{E_0(X_a)}{\sqrt{2\alpha}\;\; \varepsilon \;\parallel \delta U_\eta \parallel^2_{L^2(I)}}.$$

**[NUM]** Outer exponent 2.996, 3.004, 3.008 for $m = 4, 5, 6$, with $\kappa_b$ within 2.5% of the closed form and independent of $\varepsilon$ and of the amplitude of $\chi$; inner exponent 4.004 and 5.000 for $m = 4, 5$, fitted inside the crossover $|X - X_a| \lesssim \varepsilon^{1/(m-3)}$, outside which the inner corner also *looks* cubic. **The earlier wedge is withdrawn**: $B_{01}$ is not proportional to $\chi$.

These are laws of a truncation and not of the geometry, for two reasons. First, $E_2$ placed on the layer shifts $E_0 \partial_\eta w_2(X_a)$ by $-\int_I g(x)(x - X_a)x^{-1}dx$ with $g := E_0 E_2$, so the inner constant can be cancelled outright. Second, and structurally: on $l \equiv -h$ the demand is carried by axial flux, $U^{(1)} = -\big((D_X - h)w_1\big)'/h$ at $\eta = 0$, so $b_s$ and $\partial_\eta w_2$ carry **three radial derivatives of the demand**. Cancelling them through $E_2$ moves the same loss into $\partial_\eta^2 w_1$, and so on. Every finite truncation compares a power of $\eta$ against three radial derivatives per power of a function vanishing at the edge, and returns a cusp at the scaling $|\eta| \sim |\xi|^3$. The Gronwall argument that no compactly supported $\chi$ satisfies $|\chi'''| \le C\chi$ is correct, and it controls exactly the first truncation. **In this class the corner geometry is not decided by any finite jet [OPEN].**

**(e) Obstruction test on moving edges: stress carried by the circulation near the corners.** The derivative loss belongs to the edge *mechanism*, not to the system. Near each corner, let the demand be produced by the circulation profile instead of by axial flux.

> **Theorem 9.17 (fixed edges) [EXACT | TRANSCRIPTION, given (G)].** Let $Z$ be a one-sided neighbourhood of an edge $X_e$ on which, for all $|\eta| < \eta_0$: $U \equiv 0$; $E = E_{\rm pl}(X)\,\big(1 + \psi(X)\big)$ with $E_{\rm pl} \propto X^{-A}$ and $\psi$ independent of $\eta$; $\int_0^X U\, dx = (2+\alpha)\eta$; and $\Pi$ independent of $\eta$. Then on $Z$, exactly,
>
> $$W = 1 - \frac{(2+\alpha)L}{X}, \qquad b_s \equiv 0, \qquad \Sigma_q = -\frac{X(l+h)}{L} + (2+\alpha)l - D_X a - la, \qquad S_n = \eta P, \quad P := 4A\Pi + E^2.$$
>
> Assume **(G)**: $w(X_a, \eta) = w(X_b, \eta) = 0$ for $|\eta| < \eta_0$. If $\psi$ vanishes to finite order $k \ge 2$ at $X_e$ with $\lambda := D_X \log(1 + \psi) > 0$ on $Z$ — $\psi > 0$ entering the layer at $X_a$, $\psi < 0$ leaving it at $X_b$ — and, at the inner edge, $\Pi(X_a) = \Pi_{\rm pl}(X_a)$, then
>
> $$w_1 = 2\lambda\,\big(1 + O(\xi)\big), \qquad \frac{|w_2|}{w_1} = O\big(|\eta|\,\xi^2\big)$$
>
> uniformly in $|\eta| < \eta_0$, and $M_0, M_1, M_2 > 0$ on $Z \times (-\eta_0, \eta_0)$ after shrinking $Z$. **The edge is fixed (Geometry I).**

*Proof.* Under the hypotheses $H_c(\log E)_\eta = 0$ and $A_X(U) = (2+\alpha)\eta/X$, so $XW = X - (2+\alpha)(2D\eta^2 + d) = X - (2+\alpha)L$; the displayed $\Sigma_q$ and $S_n$ then follow from (4.7) and (4.9), and $b_s = 2D_X U/E = 0$. Theorem 9.4 integrates from the edge using (G): $Hw_1 = \pm \int H\Sigma_q\, dx/x$ and $Ew_2 = \pm(\eta/L)\int P\, dx$. As $\xi \to 0$ the term $2D_X\lambda$ dominates $\Sigma_q$, giving $w_1 \sim 2\lambda \sim 2X_e\psi'$. Since $4A\Pi_{\rm pl} + E_{\rm pl}^2 = 0$

, $P = 4A(\Pi - \Pi_{\rm pl}) + E_{\rm pl}^2(2\psi + \psi^2)$, and with $\Pi(X_a) = \Pi_{\rm pl}(X_a)$ (automatic at $X_b$) this is $O(\psi)$; so $Ew_2 = O(\eta\,\xi\psi)$ and $|w_2|/w_1 = O(|\eta|\xi\psi/\psi') = O(|\eta|\xi^2)$. With $b_s = 0$, $M_2 = a^3\left[2w_1^2 - (a-2)w_2^2\right]$, and $a - 2 = \alpha - 2\lambda > 0$ for small $\psi$. ▫

**[NUM]** $\alpha = 0.2$, $I = [20, 60]$, $k = 4$. Circulation-driven zones $(20, 21)$ and $(59.5, 60)$, a $U$-driven plateau between them with $\varepsilon = 0.3$, and two amplitudes solved for $c_Q = 0$ and $\Pi(X_a) = \Pi_{\rm pl}(X_a)$ (residuals $\leq 3 \times 10^{-12}$). At $\eta = 0$: $w_1 > 0$ on the whole open layer, $\min M_0 = 0.25$, $\max|\lambda| = 0.043$. On both zones $\times\,|\eta| \leq \frac{1}{2}$, at $41 \times 3 \times 10^5$ points: $M_0, M_1, M_2 > 0$ everywhere, minimum angular margin 0.056 rad (inner) and 1.19 rad (outer), $|w_2|/w_1 \propto \xi^{1.94}$ and $\xi^{2.00}$. The corner quadratures reproduce the global $w_1$ at $\eta = 0$.

**Status of (G) [COND].** Four scalar functions must vanish identically in $\eta$: $w_1(X_a, \eta)$, $Ew_2(X_a, \eta)$, $c_Q(\eta)$ and $c_N(\eta)$ — even, odd, even, odd under parity. At first order the odd pair is solved exactly by placing $E_2$ inboard and in two bumps on the plateau with $\int g\,dx/x = 0$, which keeps $\Pi_2 \equiv 0$ on both zones (residuals $10^{-16}$ relative; the inboard $E_0$-weighted integral of $E_2$ comes out positive, as Corollary 9.16 requires). Beyond first order (G) rests on height-parametrised versions of Theorems 9.14 and 9.6, and is not proved here.

**The cost, named.** On $Z$, $\alpha_{\rm loc} = \alpha - 2\lambda \neq \alpha$: the stationary relation of §9.16.2 is relaxed exactly where the edge is fixed. Whether fixed edges survive with $\alpha_{\rm loc} \equiv \alpha$ held up to the edge is the undecided question of (d).

**(f) What moves.** Two boundaries must be kept apart: the **mean matching layer**, where $\alpha_{\rm loc} = \alpha$, and the **support of the demanded stress**. The truncation laws of (d) move the second non-smoothly — $X_b(\eta) = X_b - (|\eta|/\kappa_b)^{1/3}$, which no material boundary of a smooth profile can do — while nothing forces the first to move. Theorem 9.17 fixes the second by pulling the first inward. The earlier single "layer endpoint" flattened the two, which is the error rule R1 exists to prevent.

**(g) The obstruction the corner exposed [EXACT | TRANSCRIPTION].** The inner corner cannot be analysed without looking inboard of $X_a$, and there the midplane base fails.

> **Theorem 9.18 (the midplane closure is local).** Let $(E, U, \Pi)$ be smooth and axis-regular with $U(\cdot, 0) \equiv 0$, and $l(X, 0) < 0$ for some $X$. Then the demanded stress at $\eta = 0$ is not, at every radius, either zero or strictly cone-admissible.

*Proof.* Suppose it were. With $U(\cdot, 0) \equiv 0$, $b_s \equiv 0$, so $s = (a, 0)$ with $a = 2 - 2l$ and $M_0 = a(a-2) = -4l(1-l)$. A nonzero demand is admissible only if $a > 0$ and $M_0 > 0$ (Theorem 9.1), i.e. only where $l < 0$. Axis regularity gives $l(0^+) = 1$. Let $X_0 := \inf\{X\colon\ l(X, 0) < 0\} > 0$. On $(0, X_0)$, $l \geq 0$, so the demand vanishes: $w_1 \equiv 0$, hence $\Sigma_q \equiv 0$, which is the layer equation (Z); by continuity it holds at $X_0$, where $l(X_0) = 0$. Lemma 9.12 then gives $l'(X_0) = h/2 > 0$, so $l > 0$ on a right neighbourhood of $X_0$, contradicting the definition of $X_0$. ▫

**[NUM]** For the base of §9.16.4: $l$ first crosses zero at $X_0 = 2.44$, where $w_1 = -2.9$; on $\{l \geq 0\}$ the demand reaches $|w_1| = 2.9$ where no cone exists, and on $\{l < 0,\ X < X_a\}$ it lies in $[-3.9, 0]$, the antipodal half-plane.

Consequences, each at its own strength. **(i)** Theorem 9.15 stands exactly as stated — strict membership on the open active layer — but it is a closure of the *layer*. The reading "Lemma $H''$ holds on the midplane fibre" is withdrawn: local closure was mistaken for global closure. **(ii)** The reflection-symmetric class is **excluded** for the global problem, since $U$ odd forces $U(\cdot, 0) = 0$. **(iii)** The contradiction re-enters the premise stack at its earliest revisable node, the choice $v \equiv 0$ of §9.16.2. A global midplane profile needs $|b_s| > 2\sqrt{l(1-l)}$ wherever the demand is nonzero and $0 \leq l < 1$ — **a nonzero midplane axial velocity inboard of the layer.** That is the design choice the source construction makes, for its stated reason: reflection symmetry would leave no radial shear of the axial velocity at $z = 0$. **(iv)** This is the correctly scoped form of §9.16.7(ii): $v = 0$ is allowed on the active layer, and $v \not\equiv 0$ is required inboard. **(v)** Theorem 9.17 assumes no reflection symmetry and imposes $U \equiv 0$ only near the corners, so its hypotheses are compatible with that class.

**(h) Freedom table.**

| jet | status |
|---|---|
| $E_0$ on the layer | $\propto X^{-A}$ in the Theorem 9.13 base; free in Theorem 9.17 zones through $\psi$, at the price $\alpha_{\mathrm{loc}} \neq \alpha$ there |
| $v = U(\cdot, 0)$ | zero under parity; **must be nonzero inboard** for the global problem (Theorem 9.18) |
| $U^{(1)}$ | total $= 2 + \alpha$ (Cor. 9.8); on the layer $= \varepsilon\, \delta U_\eta$, fixed by the demand wherever $l \equiv -h$ |
| $\partial_\eta E\|_0$ | **forced to zero** by $w_2(\cdot, 0) \equiv 0$ on the $v = 0$ layer/corner midplane, in either parity class; an **independent** midplane state once $v \neq 0$ (Prop. 9.19), and the control that closes the global midplane (§9.16.9(h)) |
| $U^{(2)} = \partial_\eta^2 U\|_0$ | zero under parity, free in [GENERAL]; enters only $\partial_\eta w_1$ |
| $E_2 = \partial_\eta^2 E\|_0$ | free in shape, but its $E_0$-weighted integral is **fixed and positive** (Cor. 9.16); reaches $B_{01}$ through $\Pi_2$ |
| $\Pi_0,\ \Pi_2$ | determined by $E_0$ and $E_0 E_2$; normalisation at radial infinity forced by the exterior |
| $\chi,\ \varepsilon,\ \psi$ | free; $\chi$ alone cannot fix an edge at any finite truncation, $\psi$ can (Theorem 9.17) |

> **Next theorems. (1)** The global midplane — closed by §9.16.9(h), Theorems 9.25–9.26. **(2)** Condition (G); its first test is settled below, and it changes the framework. **(3)** Whether a fixed edge exists with $\alpha_{\mathrm{loc}} \equiv \alpha$ held up to the edge: a corner-layer problem in the variable $\eta/\xi^3$, not a Taylor problem.

**Height continuation is marching, not initial-value continuation [EXACT | TRANSCRIPTION + NUM].**

*The rank of (G).* Outboard of $X_b$ the exterior — $U = 0$, $E \propto X^{-A}$, $\Pi$ independent of $\eta$, all axial flux $(2 + \alpha)\eta$ inboard — is stress-free for every $\eta$ (proof of Theorem 9.17). So $w_1 H$ and $E w_2$ are constant there, and $c_Q(\eta) = L\, H\, w_1(X_b, \eta)$ and $c_N(\eta) = L\, E\, w_2(X_b, \eta)$. The pair $c_Q = c_N = 0$ *is* the pair $w(X_b, \eta) = 0$ **[REDUNDANT]**. Likewise $w(X_a, \eta) = 0$ holds automatically as long as the inboard region stays stress-free, since $w$ is a radial integral of the sources from the axis. The general endpoint/moment system therefore has **rank four**. Within the stress-free inboard class it is **two scalar outer-edge conditions plus one functional stress-free PDE constraint**: keep the inboard region stress-free, and satisfy the two identities at $X_b$. Those two the layer's own perturbation directions can supply — Theorem 9.13's mechanism for $c_Q$, and the triangular direction of Theorem 9.6 for $c_N$.

*The inboard equations march.* Put $\varepsilon := \log E$ and $H_c := D\eta + dU$, the similarity axial velocity. Solving $\Sigma_q = 0$ and $\Sigma_n = 0$ for the $\eta$-derivatives gives

$$H_c\, \varepsilon_\eta - d\, l\, A_X(U_\eta) = \frac{2L\, D_X^2 \varepsilon - L\, l\, a}{X} - l\big(1 - 2D\eta A_X(U)\big) - h(1 - 2\eta U),$$

$$H_c\, U_\eta + d\, \Pi_\eta - d\, (D_X U)\, A_X(U_\eta) = \frac{2L\, D_X^2 U}{X} - \big(1 - 2D\eta A_X(U)\big) D_X U - A(1 - 2\eta U)U + 4A\eta\Pi + 2\eta D_X \Pi.$$

The $\eta$-derivatives enter only through $H_c\ (\varepsilon_\eta, U_\eta)$ and through radial integrals of them. The highest radial derivatives are $2L\, D_X^2/X = 2LX\, \partial_X^2 + \ldots$.

**Theorem 9.27 (height marching).** The vanishing-residual equations form a nonlocally coupled parabolic system in $\eta$,

$$\partial_\eta(\varepsilon, U) = \frac{2LX}{H_c}\,\partial_X^2(\varepsilon, U) + \text{lower order},$$

forward in the direction of the axial velocity $H_c$. At every order the new jets solve the operator of Theorem 9.25, with the same coefficients $(l_0, v, E_0)$, forced by $2X\partial_X^2$ of the previous jets. At second order in $\eta$, writing $\varepsilon = \varepsilon_0 + \eta\varepsilon_1 + \eta^2\varepsilon_2/2$ and likewise for $U$ and $\Pi$, with $U_0 = v$ and $l_0 = \frac{1}{2} + X\varepsilon_0'$:

$$v\,\varepsilon_2 - l_0\,A_X(U_2) = 2X\varepsilon_1'' + \left(2 - X + XA_X(U_1) + 4X\varepsilon_0'\right)\varepsilon_1' + \left(h - \tfrac{1}{2} - U_1\right)\varepsilon_1 + 2D\,l_0\,A_X(v) + 2hv,$$

$$v\,U_2 + \Pi_2 - (D_X v)\,A_X(U_2) = 2XU_1'' + \left(2 - X + XA_X(U_1)\right)U_1' - U_1(1 + U_1) + 2D\,A_X(v)\,D_X v + 4A\Pi_0 + 2D_X\Pi_0 + 2Av^2,$$

$$\Pi_2 = -\int_X^\infty \frac{E_0^2\,(\varepsilon_2 + 2\varepsilon_1^2)}{y}\,dy.$$

*(Extracted symbolically from (4.8)–(4.9).)* **[NUM]** On the constructed lift ($v_0 = 3$), high-frequency perturbations of $v$ and of $\log E$ near $X = 4$ give $\delta U_1/[(2X/v)\,\delta v''] = 1.006,\ 0.994,\ 0.999$ and $\delta e_1/[(2X/v)\,\delta\varepsilon''] = 1.009,\ 0.995,\ 0.999$ for wavenumbers $15, 30, 60$. The symbol and its sign are those of the theorem.

Consequences, each at its own strength.

**(i) The formal hierarchy [FORMAL].** No order introduces a new operator, so there is no new resonance. Under (NR) every order is uniquely solvable, provided the forcing keeps the tameness that Theorem 9.26 uses at $X_s$ — not proved here for orders beyond the first. The four identities do not overdetermine the second-order jets.

**(ii) Derivative loss [EXACT].** Each order costs two radial derivatives and a factor $2X/H_c$, which near the return point, where $v \to 0$, is a factor $1/v$. The hierarchy has the principal part of the heat equation, whose formal power series in time diverges for generic smooth — and even analytic — data. This blocks convergence, not merely truncation: the formal series is not a construction.

**(iii) The midplane is not a Cauchy surface [FAIL].** With $v > 0$ inboard, continuation to $\eta < 0$ is backward-parabolic, and a backward-parabolic problem has no smooth solution for generic smooth data and no continuous dependence. The marching direction also reverses on the stagnation surface $H_c = 0$, at $\eta_*(X) \approx -v/(D + U_1)$. Beyond $X_s$ that surface is the midplane itself, since there $H_c = D\eta$; and as $X \to X_s$ from inside, $\eta_* \to -v/D \to 0$. **[NUM]** For $v_0 = 3$, $\eta_* \approx +0.40$ to $+0.53$ on the plateau, $-6.9 \times 10^{-2}$ at $X = 12$ and $-3.7 \times 10^{-5}$ at $X = 13.5$, with smallest positive value $5 \times 10^{-8}$. The bound $1/\,\|\,e_1\,\|_\infty = 3.6 \times 10^{-4}$ is not the binding scale. So no $\eta_0 > 0$ exists for which the closed midplane, taken as generic Cauchy data, continues to a stress-free inboard region on both sides. The premise this contradicts is the choice of the midplane as the design surface, and that is the premise re-entered.

*What survives.* Theorems 9.25–9.26 remain true as statements about a slice: every slice of a stress-free marching solution satisfies the lift relations, and (NR) is an open condition along slices. What does not survive is using the designed midplane as initial data. On the layer, where the stress is a free design variable, none of this applies. Margins persist by openness in $C^1$, and the edges are Theorem 9.17's problem. The zone between $X_s$ and $X_a$, exactly stress-free for every $\eta$ but characteristic at $\eta = 0$ ($H_c = D\eta$), is where the marching solution must hand over to the layer, and whether it can stay exterior-type is part of the open problem.

**Obstruction tests.** *(A) Second-order incompatibility:* no — the system above is uniquely solvable by the Theorem 9.25 operator. *(B) Derivative loss:* yes, in the precise sense of (ii)–(iii) — it rules out the initial-value framework, not (G). *(C) Positivity:* $|\eta| < 1/\,\|\,e_1\,\|_\infty$ is not the binding constraint. *(D) Cone closure:* no

loss on the open active layer; inboard there is no margin to lose, which is why the inboard region must stay exactly stress-free. *(E) Edge conflict:* open, located at the characteristic zone $(X_s, X_a)$. *(F) Resonance:* none new at any order, and (NR) is open along smooth continuation.

**The stagnation surface is not a design manifold [EXACT | TRANSCRIPTION + NUM].** The marching direction reverses on $\Sigma_* := \{H_c = 0\}$, so the natural candidate for a two-sided design surface is $\Sigma_*$ itself. It fails for three reasons, each proved at its own strength.

*The equations on $H_c = 0$.* Setting $H_c = 0$ in the marching equations, without dividing by it, gives at $(X, \eta_*(X))$

$$-d\, l\, A_X(U_\eta) = R_q, \qquad d\,\Pi_\eta - d\,(D_X U)\, A_X(U_\eta) = R_n, \qquad U_* = -\frac{D\,\eta_*}{d},$$

where $R_q, R_n$ are the right-hand sides of the marching equations. The trace of $U$ is fixed by the shape of $\Sigma_*$. The remaining $\eta$-derivatives enter only through $A_X(U_\eta)(X, \eta_*) = X^{-1}\int_0^X U_\eta\big(x, \eta_*(X)\big)\, dx$ and $\Pi_\eta$, which are integrals along the **horizontal** slice through the point. And $R_q, R_n$ contain $\partial_X^2$ at fixed $\eta$, which on a curve $\eta = \eta_*(X)$ equals the second derivative of the trace minus $2\eta_*' u_{X\eta} + \eta_*'^2 u_{\eta\eta} + \eta_*'' u_\eta$ (verified symbolically). **Unless $\eta_*$ is locally constant, these are not equations for a trace on $\Sigma_*$**: they couple each point of $\Sigma_*$ to the solution off it.

*Shape and orientation.* Where $\partial_\eta H_c = D + dU_\eta - 2\eta U \neq 0$, $\Sigma_*$ is a smooth graph with

$$\eta_*' = -\frac{d\,U_X}{D + dU_\eta - 2\eta U}\Big|_{\eta = \eta_*}.$$

The forward direction is $\operatorname{sign} H_c$, and $H_c \approx \partial_\eta H_c\,(\eta - \eta_*)$. So both sides march away from $\Sigma_*$ exactly when $\partial_\eta H_c > 0$ — a source — and both march into it when $\partial_\eta H_c < 0$ — a sink. The model $\beta\eta\, u_\eta = c\, u_{XX}$ separates in radial modes as $|\eta|^{-c\lambda/\beta}$, with $\lambda > 0$ the modal eigenvalue (verified symbolically). At a source these modes are singular, so a smooth solution carries **no free modes** and is fixed by its radial data. At a sink they are the non-smooth modes $|\eta|^{c\lambda/|\beta|}$ decaying into $\Sigma_*$, and smooth matching across it is overdetermined. At the ends of the strip $d = 0$ and $H_c(X, \pm 1) = \pm D$, so **both ends are outflow**. For every $X$ with $v(X) > 0$, $H_c$ rises from $-D$ at $\eta = -1$ to $v$ at $\eta = 0$, so a source lies somewhere in $(-1, 0)$ — the downward-displaced stagnation level that the source construction's axial bias produces.

> **Proposition 9.28 (stagnation geometry).** The stagnation set carries no intrinsic trace problem where it is not horizontal. Where it is horizontal it is a Fuchsian source or sink according to the sign of $\partial_\eta H_c$, and a smooth solution at a source has no free functional data. The strip $|\eta| < 1$ has outflow at both ends.

> **Theorem 9.29 (no horizontal stagnation slice).** Suppose $H_c \equiv 0$ on a slice $\eta = \eta_0$, $|\eta_0| < 1$, over the radii where $l(\cdot, \eta_0) \geq 0$, and $l(\cdot, \eta_0) < 0$ somewhere. Then the demand on that slice is not, at every radius, either zero or strictly cone-admissible.

*Proof.* $H_c = 0$ gives $U(\cdot, \eta_0) \equiv -D\eta_0/d$, so $b_s = 2D_X U/E = 0$ and $s = (a, 0)$; a nonzero demand is then admissible only where $l < 0$. Axis regularity gives $l(0^+, \eta_0) = 1$. Let $X_0 := \inf\{l < 0\}$. On $(0, X_0)$ the demand must vanish, so $\Sigma_q \equiv 0$ there. At $X_0$, where $l = 0$ and $H_c = 0$, the relation $\Sigma_q = 0$ reads $l' = h/(2d) > 0$ (verified symbolically), contradicting the definition of $X_0$. ▫

This is Theorem 9.18 at height $\eta_0$; Theorem 9.18 is the case $\eta_0 = 0$, $v \equiv 0$. It excludes a horizontal stagnation slice through the inboard region — which is exactly the case in which $\Sigma_*$ would be a legitimate trace manifold. The one horizontal piece that does exist, the midplane for $X \geq X_s$ where $H_c = D\eta$, lies where $l = -h < 0$ and is untouched.

*At the handover point.* $H_c = v + \eta(D + U_1) + \eta^2(U_2/2 - v) + \ldots$, and $v$ is flat at $X_s$. Hence $\eta_* = -v/(D + U_1) + O(v^2)$ meets the midplane **infinitely flatly** from below, with $\partial_\eta H_c \to D > 0$. The meeting is a $C^\infty$ graph and a source, but it is also a change of type: from a non-horizontal interface to a horizontal Fuchsian source.

*The closed midplane as a slice.* **[NUM]** For $v_0 = 3$ the local stagnation level on the plateau is a sink at $\eta_* \approx +0.40$ to $+0.53$, where $D + U_1 \approx -6$, and the source feeding the midplane there lies in $(-1, 0)$ at $O(1)$ depth. Near $X_s$ the stagnation level is a source lying flat against the midplane. The midplane slice therefore cannot arise from local marching off a stagnation surface near $\eta = 0$; it can only be a slice of a global solution.

**Obstruction tests.** *(A) Trace overdetermination:* yes — $\Sigma_*$ carries no free functional data, having no intrinsic trace problem where it is not horizontal and none free at a horizontal source. *(B) Two-sided orientation:* partial — sources and sinks both occur along the local stagnation set of the closed design. *(C) Transversality:* holds near the midplane (near $X_s$, $\partial_\eta H_c \to D$); it fails only where $D + U_1 = 0$, where $\eta_*$ leaves the neighbourhood. *(D) Meeting at $X_s$:* smooth but flat, with a change of type; not an obstruction by itself. *(E) Midplane trace realization:* locally impossible on the plateau; it is a global question. *(F) Layer rank:* no failure — the angular direction of Theorem 9.13 and the triangular direction of Theorem 9.6 stay independent for small $\eta$ by openness **[COND on Theorem 9.6]**.

**What is the system, and what is its clock? [EXACT | TRANSCRIPTION].** Before posing a boundary-value problem across $\Sigma_*$, R1 and R15 require asking whether $\{H_c > 0\}$ and $\{H_c < 0\}$ are one evolutionary subsystem or two, and whether $\eta$ is their clock at all.

*The intrinsic clock.* Where $H_c \neq 0$ put $\tau_\eta = 1/H_c$, and write $u(X, \eta) = \tilde{u}\big(X, \tau(X, \eta)\big)$. Then, exactly (verified symbolically),

$$\partial_\eta|_X = \frac{1}{H_c}\,\partial_\tau, \qquad \partial_X|_\eta = \partial_X|_\tau + \tau_X\,\partial_\tau, \qquad \partial_X^2|_\eta = \partial_X^2 + 2\tau_X\,\partial_X\partial_\tau + \tau_X^2\,\partial_\tau^2 + \tau_{XX}\,\partial_\tau,$$

so the principal part of the stress-free system becomes

$$H_c\,u_\eta - 2LX\,u_{XX} = \partial_\tau\tilde{u} - 2LX\big(\partial_X + \tau_X\partial_\tau\big)^2\tilde{u} - 2LX\,\tau_{XX}\,\partial_\tau\tilde{u}.$$

Written without coordinates this is $2LX\,V^2 - T$, with the diffusion field $V = \partial_X|_\eta$ and the drift field $T = H_c\partial_\eta = \partial_\tau|_X$.

> **Proposition 9.30 (intrinsic clock).** (i) On each sign-domain of $H_c$ there is a time function — $\eta$ where $H_c > 0$, $-\eta$ where $H_c < 0$ — constant along $V$ and increasing along $T$. Each domain is therefore **forward-parabolic along its own drift**, and "backward in $\eta$" is an orientation artifact. (ii) Along drift lines every zero of $H_c$ lies at infinite intrinsic depth: $\tau \simeq \kappa^{-1}\log|\eta - \eta_*|$ at a simple zero, $\tau \sim (\eta - \eta_*)^{1-m}$ at a zero of order $m$, and faster at a flat zero. A source ($\kappa > 0$) is at $\tau = -\infty$ on both sides; a sink ($\kappa < 0$) is at $\tau = +\infty$. (iii) $\tau$ is **not** a Cauchy clock: $V\tau = \tau_X \simeq -\eta_*'/\big(\kappa(\eta - \eta_*)\big)$, which diverges at $\Sigma_*$ unless $\eta_*' = 0$. The Cauchy slices of the operator remain the diffusion leaves $\eta =$ const.

*Does any finite amount of evolution cross $\Sigma_*$?* Along the drift, no. Along the diffusion, yes, exactly where $\Sigma_*$ is transverse to the leaves. A leaf $\eta = \eta_0$ meets a non-horizontal $\Sigma_*$ at the radius where $\eta_*(X) = \eta_0$. There the drift $H_c\partial_\eta$ vanishes but the radial operator $2LX\partial_X^2$ does not, and the nonlocal terms $A_X(U_\eta)$ and $\Pi_\eta$ integrate along the same leaf. The two sign-domains are joined on that leaf at finite radial distance by an ordinary second-order radial equation.

> **Theorem 9.31 (when the sign-domains separate).** (a) Where $\Sigma_*$ is horizontal, no leaf crosses it. The two sides are **separate boundary-value problems**, each forward along its own drift, meeting only at infinite intrinsic depth. At a source the model modes $|\eta - \eta_*|^{-c\lambda/\kappa}$ grow as $\tau \to -\infty$ and are excluded by

> boundedness, so smoothness across $\Sigma_*$ requires **only equality of the asymptotic traces**; all higher jets then agree, because the Fuchsian recursion is the same on both sides. At a sink the modes $|\eta-\eta_*|^{c\lambda/|\kappa|}$ decay with non-integer exponents, and smoothness requires **every** such coefficient to vanish — overdetermined. (b) Where $\Sigma_*$ is transverse to the leaves, the two sides are **one boundary-value problem**. The closure relation is the radial equation along each crossing leaf, which is non-degenerate at the crossing and enforces continuity of the fields and their radial fluxes; the neighbouring points of one leaf carry drift of opposite orientation, so the forward–backward character is **invariant** there (Outcome B), as in the classical $x\,u_t = u_{xx}$.

*Classification of* $\Sigma_*$*.* It carries no state (R15: it is the zero set of a reconstructible field) and no intrinsic trace (Proposition 9.28). Where horizontal, it is an **asymptotic boundary** — an initial state at $\tau=-\infty$ for a source, a final state at $\tau=+\infty$ for a sink. Where transverse, it is a **Fichera-type interior degeneracy** of the drift, crossed by the diffusion, whose matching is enforced by the equation rather than imposed as independent transmission conditions.

*The inboard verdict.* Theorem 9.29 excludes a horizontal $\Sigma_*$ through the inboard region, so part (a) never applies there. **[NUM]** For the closed design ($v_0=3$), $D+U_1$ passes through zero between the plateau and the return. So $\eta_*=-v/(D+U_1)$ runs from $-\infty$ to $0^-$ on $(X_{\text{pole}},X_s)$, and **every leaf with** $\eta<0$ **crosses** $\Sigma_*$ **there**. That joins the lower outer lobe $\{\eta<0,\ X\geq X_s\}$, where $H_c=D\eta<0$, to the inboard region, where $H_c>0$. **The intrinsic-clock decomposition fails inboard [FAIL]**, and the two sign-domains form one boundary-value problem. The upper outer lobe is joined to the inboard region without any crossing. Part (a) applies only to the horizontal piece of $\Sigma_*$, the midplane for $X\geq X_s$, where it gives the upper and lower lobes their common trace — but each lobe is still attached to the one inboard system along its leaves.

*The global boundary-value problem.* It is therefore **[EXACT] invariantly the correct inboard problem**, not merely the manuscript's current endpoint. The clock analysis sharpens it: the forward/backward labels inside each domain carry no content; the content lies at the leaf crossings of $\Sigma_*$ (interior, equation-enforced matching), the **sinks** (overdetermined smoothness), the **tangencies** $\eta_*'=0$ (where a leaf touches $\Sigma_*$ and the two regimes of Theorem 9.31 meet), and the **corner** $(X_s,0)$, where $\Sigma_*$ turns horizontal infinitely flatly.

*Condition (G) and the layer.* Nothing here changes (G) beyond the inboard constraint. It remains two outer-edge conditions at $X_b$ plus one functional stress-free constraint. The layer sits at $X\geq X_a\geq X_s$, where $\Sigma_*$ is the horizontal midplane, so within the layer's radial range the upper and lower branches are separate and joined by their common midplane trace. The layer needs only the physical handover trace $(E,U)(X_a,\eta)$, one branch per sign of $\eta$ — not a continuation through $\Sigma_*$ in its own region. That trace, however, comes from the single inboard boundary-value problem.

**Obstruction tests.** *(A) Clock artifact fails:* yes where $\Sigma_*$ is transverse to the leaves — the forward–backward character is invariant there (Theorem 9.31(b)); no inside each sign-domain (Proposition 9.30(i)). *(B) Infinite depth fails:* no along drift lines; yes along the diffusion leaves, which cross $\Sigma_*$ at finite radial distance. *(C) Splitting into separate problems fails:* yes inboard — the closure relation is the non-degenerate radial equation on crossing leaves. *(D) Asymptotic matching overdetermines:* at sinks, yes; at horizontal sources, no — equality of traces suffices. *(E) Radial derivatives destroy the clock picture:* yes — $\tau_X\to\infty$ at a non-horizontal $\Sigma_*$. *(F) Layer requires global continuation:* no within its own radial range; it needs the adjacent traces, which are produced by the inboard boundary-value problem.

**Which singularities are forced? [EXACT | TRANSCRIPTION + NUM].** The singular set is the sinks, the tangencies of $\Sigma_*$ with the leaves, and $(X_s,0)$. Before any existence theory, one must ask which of these a design can remove.

*The conditions.* From $H_c=D\eta+dU$ we have $\partial_\eta H_c=D+dU_\eta-2\eta U$ and $\partial_X H_c=d\,U_X$. On $\Sigma_*$, where $U=-D\eta/d$,

$$\text{source} \Leftrightarrow \frac{D(1+\eta^2)}{d} + d\,U_\eta > 0, \qquad \text{leaf tangency} \Leftrightarrow U_X = 0, \qquad \eta_*' = -\frac{d\,U_X}{\partial_\eta H_c}.$$

Both are conditions on the direction of the normal $\nabla H_c$ along $\Sigma_*$. A sink is a normal pointing downward, and a leaf tangency is a vertical normal.

> **Proposition 9.32 (stagnation Gauss map).** Let $\theta$ be the angle of $\nabla H_c$ from the upward vertical along $\Sigma_*$. Then $\Sigma_*$ is a single transverse source curve iff $\theta$ stays in one open quadrant, $(0, \pi/2)$ or $(-\pi/2, 0)$. Sinks are the crossings of $|\theta| = \pi/2$ and leaf tangencies the crossings of $\theta = 0$, so **R15 merges them**: they are the two walls of one condition on one primitive object, the oriented curve $\Sigma_*$. Topologically, $H_c(X, \pm 1) = \pm D$ gives $\#\{\text{sources}\} - \#\{\text{sinks}\} = 1$ on every vertical line, so a single source per line is admissible and **no sink is forced**. An interior leaf tangency is forced by Rolle only if $\eta_*$ takes equal values at two points of one source branch.

> **Theorem 9.33 (forced singular points).** (a) *Axis.* $(2L/X)D_X^2 = 2L(X\partial_X^2 + \partial_X)$ is the two-dimensional radial Laplacian in $r \propto X^{1/2}$, and Cartesian regularity makes $H_c$ smooth in $r^2$. So $\partial_r H_c = 0$ at $r = 0$. $\Sigma_*$ must meet the axis, since $H_c(0, \cdot)$ runs from $-D$ to $D$, and it meets it in a **fold tangency** with the leaves: $\eta_*(r) \simeq \eta_*(0) + \gamma r^2$, with local model $(\kappa t + \gamma r^2)\,u_t = c\,\Delta_r u$. (b) *Corner.* Every height jet $v, U_1, U_2, \ldots$ of the lift is flat at $X_s$ and vanishes beyond it, so $H_c = D\eta + F$ with $F$ flat at $\xi = X - X_s \to 0^-$ and $F \equiv 0$ for $\xi \geq 0$. $\Sigma_*$ joins the horizontal exterior piece **infinitely flatly**, and in the straightened variable $t = \eta - \eta_*(\xi)$ the operator is $D\,t\,u_t = c\,u_{\xi\xi}$ plus flat terms — the horizontal-source model of Theorem 9.31(a) to all orders. Hence the singular set cannot shrink to $\{(X_s, 0)\}$: **the axis fold and the flat corner are both forced**, and everything else is design-dependent.

*What controls the geometry.* Inboard, $(E, U)$ is the output of a determined system: two equations for two fields, second order in $X$ with axis regularity and a handover trace, and no data in $\eta$ because both ends are outflow. The jets $U_\eta$, $U_{\eta\eta}$, $E_\eta$, $E_{\eta\eta}$ and $\Pi_\eta$ are therefore **not independent design variables**. The geometry of $\Sigma_*$ is state of the solution (R1), and the only freedom is the handover trace $(E, U)(X_a, \eta)$ — two functions of $\eta$. The geometry-control map is $\mathcal{G}' = (\delta U,\ \delta U_\eta)|_{\Sigma_*} \circ \mathcal{S}'$, where $\mathcal{S}'$ is the linearised solution operator of the inboard problem. The component $\delta(\partial_X H_c)$ is not independent: it is reconstructed from the tangential derivative of $\delta U|_{\Sigma_*}$. So the map is functional, from functions of $\eta$ to functions on $\Sigma_*$, and its range cannot be computed before the linearised existence theory. "Geometry first, existence second" separates only locally: near the midplane, where the jets of the lift control $\Sigma_*$; at the axis, by symmetry; and at the corner, by flatness.

*Near the midplane: sinks in every tested lift.* **[NUM]** Where $|\eta_*| = |v/(D + U_1)| < 0.05$ the linear stagnation point is reliable at leading order. In six closed-midplane lifts ($v_0 = 3$; return of $v$ on $[11, 14]$, $[11, 20]$, $[11, 30]$; or decay to $v_r = 0.05, 0.01, 0.002$ before a flat return) there are near-midplane **sinks** in every case. They move with the return, to $X \in [13.64, 13.83]$, $[18.8, 19.4]$, $[27.5, 28.7]$, $[15.2, 19.7]$, $[15.1, 19.6]$ and $[15.0, 17.2]$ respectively. The mechanism is the degenerate-zone asymptotics $U_1 \simeq -(2X^3/hE_0^2)(vv''')' + \ldots$. Its coefficient grows like $X^{4.2}$, so $|U_1| \gg D$ wherever $v$ varies, and $U_1 < 0$ on convex parts of the decay. Lengthening the return lowers $\max|U_1|$ only like $\ell^{-1.5}$ to $\ell^{-2}$ (3332, 363, 120), and lowering the return amplitude steepens the decay instead (62, 99, 189). This is evidence that the lift makes sinks generic, not a proof that they are forced. Test C is neither confirmed nor refuted.

*Condition (G) and the handover.* A geometry correction is a change of the handover trace, two functions of $\eta$. The layer's two correction directions are also functions of $\eta$ (Theorems 9.13 and 9.6, parametrised in height), so there is **no rank mismatch**: $w(X_b, \eta) = 0$ can absorb any trace change locally **[COND on Theorem 9.6]**.

*Geometry versus existence — the scalar estimate.* Multiplying the principal scalar equation $H_c u_\eta - 2L(Xu_X)_X = f$ by $u$, with $u(X_a, \eta) = 0$ after the handover trace is subtracted, gives

$$\int\int 2LX\,u_X^2+\frac{D}{2}\int\big(u(X,1)^2+u(X,-1)^2\big)\,dX-\frac{1}{2}\int\int\partial_\eta H_c\,u^2=\int\int f\,u,$$

and the leafwise Hardy bound $u(X)^2\le\log(X_a/X)\int_X^{X_a}x\,u_x^2$ makes **the scalar model** coercive when

$$\sup_{|\eta|<1}B(\eta)<4(1-\alpha),\qquad B(\eta):=\int_0^{X_a}\big(\partial_\eta H_c\big)_+\log\frac{X_a}{X}\,dX.$$

*Test A* **[EXACT]**. At every radius $\int_{-1}^{1}(\partial_\eta H_c)_+d\eta\ge\int_{-1}^{1}\partial_\eta H_c\,d\eta=2D$, and $\int_0^{X_a}\log(X_a/X)\,dX=X_a$. So $\sup_\eta B\ge\frac{1}{2}\int_{-1}^{1}B\,d\eta\ge DX_a$. With $D=(1-\alpha)/2$, **(NR$_\eta$) forces** $X_a<8$ **for every** $\alpha$. The axis concentration of Theorem 9.33(a) adds nothing, since $\log(X_a/X)$ is integrable, so Test D reduces to Test A. Neither excludes (NR), which holds in lifts with $X_s\le X_a<8$ (below).

*The full coupled system.* Linearise the two stress-free equations of Theorem 9.27 about a stress-free background. Take $(e,u)=(\delta\log E,\delta U)$ and $p=\delta\Pi=-\int_X^{X_a}E^2e\,dy/y$ after the handover trace is subtracted, with $A_X(u)=X^{-1}\int_0^X u$. Verified by computer algebra:

$$H_ce_\eta-2L(Xe_X)_X+b_e\,e_X+d\,\varepsilon_\eta u-d\,l\,A_X(u_\eta)-2D\eta\,l\,A_X(u)-2h\eta u=f_e,\qquad b_e=L(2-4l)+XW,$$

$$H_cu_\eta-2L(Xu_X)_X+XW\,u_X+\big(dU_\eta+A(1-4\eta U)\big)u-d\,XU_X\,A_X(u_\eta)-2D\eta\,XU_X\,A_X(u)+d\,p_\eta-4A\eta\,p-2\eta E^2e=f_u.$$

Here $(f_e,f_u)$ is minus the operator applied to an extension of the handover trace, so it contains the trace's $\eta$-derivative.

> **Theorem 9.34 (full energy identity; the scalar condition is not a coercivity condition) [EXACT | TRANSCRIPTION].** (a) The transport fields have divergences
>
> $$\partial_X(XW)+\partial_\eta H_c=(1+D)(1-2\eta U),\qquad\partial_X b_e+\partial_\eta H_c=(1+D)(1-2\eta U)-4L\,l_X,$$
>
> **free of** $U_\eta$. Multiplying by $(e,u)$ and integrating over $(0,X_a)\times(-1,1)$ gives
>
> $$\iint 2LX(e_X^2+u_X^2)+\frac{D}{2}\int(e^2+u^2)\big|_{\eta=\pm1}dX+\int L\,e(0,\eta)^2d\eta+\iint c_e\,e^2+\iint c_u\,u^2+\mathcal{Z}+I_1+I_2+I_3=\iint(f_ee+f_uu),$$
>
> $$c_e=-\tfrac{1}{2}(1+D)(1-2\eta U)+2L\,l_X,\qquad c_u=d\,U_\eta-\tfrac{1}{4}+\tfrac{3}{2}h-(\tfrac{1}{2}+5h)\eta U.$$
>
> $\mathcal{Z}$ collects the zero-order cross terms $d\varepsilon_\eta ue$, $-2D\eta lA_X(u)e$, $-2h\eta ue$, $-2D\eta XU_XA_X(u)u$, $-4A\eta pu$ and $-2\eta E^2eu$, and the three $\eta$**-derivative couplings** are
>
> $$I_1=-\iint d\,l\,e\,A_X(u_\eta),\qquad I_2=-\iint d\,XU_X\,u\,A_X(u_\eta),\qquad I_3=\iint d\,u\,p_\eta.$$
>
> The axis term comes from $b_e(0)=-2L$ (axis regularity gives $l=1$). The terms at $X_a$ vanish because the trace is subtracted, and at $\eta=\pm1$ the flow is outward since $H_c=\pm D$. (b) The $-\frac{1}{2}\partial_\eta H_c$ of the scalar model is an artefact of dropping the radial drift. In the system the budget depends on $U_\eta$ only through $+dU_\eta$, so **sinks consume it and sources supply it**. (c) Take bounded positive local multipliers $(\mu e,\nu u)$. The $O(\partial_\eta)$ part of $I_1+I_3$ reduces to zero order for all test profiles iff $\nu=\nu(\eta)$ and $\mu l=\nu E^2$, which is impossible across the crossing $l=0$. That of $I_2$ does so iff $U_X\equiv0$ on the leaf, because its kernel $dU_X(X)\mathbf{1}_{x<X}$ is Volterra. Otherwise tests $(\psi\cos k\eta,\ \phi\sin k\eta)$ or $\phi_1\cos k\eta+\phi_2\sin k\eta$, with fixed radial profiles, change the form by $O(k)$ of either sign while the energy norm stays $O(1)$. **Hence no bounded positive local diagonal multiplier makes the linearised form coercive, even on smooth test functions (Lions), on any leaf where** $l$ **changes sign or** $U_X\not\equiv0$**.** At $\eta=0$ both happen.

*Proof.* The linearisation and divergences are checked by computer algebra (cv22, cv23). Integrate by parts in $X$ and $\eta$. For (c), Fubini turns $I_3$ into $\iint\nu\,d\,E^2\,e_\eta\,A_X(u)$ up to zero order; pairing with $I_1$ leaves the $O(k)$ coefficient $\frac{k}{2}\int d(\nu E^2-\mu l)\,\psi A_X(\phi)$. A Volterra kernel is symmetric only if it vanishes. ▫

**Band tests: what is necessary.** Take $w=(0,\phi(X)\chi(\eta))$ or $(\psi\chi,0)$ with $\chi$ concentrated at one height. Then $I_1$–$I_3$ become zero-order and the form reduces **exactly** to radial forms. At $\eta=0$:

$$Q_u(\phi) = \int 2X\phi'^2 + \left(U_1 - \tfrac{1}{4} + \tfrac{3}{2}h\right)\phi^2 + \tfrac{1}{2}XU_1'\,\phi\,A_X(\phi), \qquad Q_e(\psi) = \int 2X\psi'^2 + \left(2l' - \tfrac{1}{2}(1+D)\right)\psi^2 + \psi(0)^2.$$

Leafwise positivity of these forms is therefore **necessary** for coercivity with unit multipliers; positive-part budgets are only sufficient for it. **[NUM]** Six closed-midplane lifts ($\lambda_{\min}$ of each form relative to $\int X\phi'^2$, Dirichlet at $X_a$, 2800 nodes):

| **lift ($l$ descent; return of $v$; $X_a$)** | $R_{\mathrm{mid}}$ | **scalar $B(0)$ vs** 3.2 | $\lambda_{\min}$ **scalar** | $\lambda_{\min}(Q_u)$ | $\lambda_{\min}(Q_e)$ |
|---|---|---|---|---|---|
| $[0.5, 10]$; $v_0 = 3$ on $[11, 14]$; 14 | 3.92 | 17.95 | $-2.81$ | $-544$ | $-5.41$ |
| same; $X_a = 16$ | 3.92 | 58.8 | $-8.35$ | $-608$ | $-6.58$ |
| $[0.2, 4.2]$; $[4.5, 7.5]$; 7.5 | 2 | 13.57 | $-1.56$ | $-128$ | $-3.08$ |
| same | 8 | 13.35 | $-1.53$ | $-188$ | $-3.08$ |
| $[0.1, 2.1]$; $[2.6, 5]$; 5 | 2 | 11.50 | $-1.01$ | $-92.1$ | $-2.53$ |
| $[0.05, 1.05]$; $[1.2, 3]$; 3 | 1.5 | 9.56 | $-0.53$ | $-52.6$ | $-1.99$ |

A longer return ($[3.8, 7.9]$) gives $B(0) = 12.06$. Only the $\eta = 0$ slice of $B_*$ is computable from the midplane jets, and it is a **lower bound**; the full $B_*$ needs the global solution. The scalar budget comes **100% from the return lobe** of $U_1 > 0$ (peaks $1.3 \cdot 10^2$–$7.9 \cdot 10^2$, log-weight $\leq 0.73$), with nothing from the axis, the plateau or the degenerate tail. Moving the handover out adds $D$log-mass ($17.95 \to 58.8$). So **the working hypothesis holds for the scalar model and fails for the system**. The scalar budget ignores the large negative $U_\eta$, but $Q_u$ is dominated by it on the axis and plateau, where the log-weight reaches 12: for the existing lift, 30.4 on $(0, 1)$ and 59.6 on $(1, X_0)$ out of 105.7. The $Q_e$ failure is partly design-free. With $l' \equiv 0$ the axis term keeps $Q_e > 0$ only for $X_a \lesssim 5$, and the descent cost $2l'$ makes every lift negative.

*Test B* fails: the budget and (NR) are structurally decoupled. With $v = \lambda\widehat{v}$ the lift obeys $(\widehat{v}\Psi' - \widehat{v}'\Psi)' = \widehat{R}_v' + \lambda^{-2}E_0^2(N_0 - l\Psi)/(X^2\widehat{v})$. So off the degenerate zone $U_1$ tends to a profile fixed by the shape of $\widehat{v}$, while $R_{\mathrm{mid}} \propto \lambda^2$ (numerically $R_{\mathrm{mid}}$ $2 \to 8$ moves $B(0)$ from 13.57 to 13.35). *Test C* gives an exact identity but no theorem. For every $x_*$,

$$B(0) \;\geq\; D\big[X_a - x_* - x_*\log(X_a/x_*)\big] + \int_{x_*}^{X_a} \frac{\Psi - \Psi(x_*)}{X}\,dX,$$

and $\Psi(X_s) = N_0/l = 2(1+h)$ is forced. With $\Psi_{\min} \in [-80, -16]$ in the lifts this gives 7–24 **[NUM]**. That every return pays is not proved: the bound depends on where $\Psi$ rises.

> **Theorem 9.35 (the height mass operator is the midplane operator on every leaf) [EXACT | TRANSCRIPTION].** Freeze $\eta$ and write the $\eta$-derivative part of the system as $\mathfrak{A}_\eta\partial_\eta(e, u)$, with $\mathfrak{A}_\eta(e, u) = \left(H_c e - d\,l\,A_X u,\; H_c u - d\,XU_X\,A_X u - d\,\mathcal{V}(E^2 e)\right)$ and $\mathcal{V}g = \int_X^{X_a} g\,dy/y$. Eliminate $e$ and put $\Psi = \int_0^X u$ and $\Phi = \Psi/H_c$. After one $X$-derivative the Schur complement is
>
> $$\mathcal{S}_\eta\Phi = (H_c^2\Phi')' + \frac{d^2E^2\,l}{X^2}\,\Phi,$$
>
> the Sturm–Liouville operator of Theorem 9.26 with $v \to H_c$ and $E_0^2 \to d^2E^2$; at $\eta = 0$, where $d = 1$ and $H_c = v$, it **is** that operator (trivially $d^2$ times it), with the same kernel, index and non-resonance condition. (a) (NR) is the midplane slice of leafwise non-resonance of the height mass operator. (b)

*Simple zeros only* ($U_X \neq 0$). At a simple zero of $H_c$ on a leaf the indicial roots are $s = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \mathrm{Ri}_*}$, where

$$\mathrm{Ri}_* = \frac{E^2\, l}{X^2\, U_X^2}\Big|_{\Sigma_*}.$$

The negative index of $\mathcal{S}_\eta$ near a simple zero is finite if $\mathrm{Ri}_* \leq \frac{1}{4}$ and infinite if $\mathrm{Ri}_* > \frac{1}{4}$, and automatically finite where $l \leq 0$. The critical value lies on the finite side. With smooth coefficients the corrections are $O(z)$, $z = X - X_*$, and in $s = -\log z$ with $\Phi = e^{s/2} w$ the equation becomes $w'' + O(e^{-s}) w = 0$, non-oscillatory by Hille–Kneser since $\int^\infty s\, e^{-s} ds < \infty$. There is finite index, but no coercivity. (b′) *Tangencies* ($U_X = 0$, $H_c$ vanishing to order $m \geq 2$ along the leaf). Here $\mathrm{Ri}_*$ does not apply. The coefficient is $\sim z^{2m}$ against a potential $q_0 = d^2 E^2 l / X^2 \neq 0$, and $\int_0 \sqrt{|q_0|}\, z^{-m} dz = \infty$; the Liouville normal form then gives infinite index if $l > 0$, and index zero locally if $l < 0$. A tangency with $l = 0$ is left open. (c) **On the leaf through the forced axis fold**, $H_c(0, \eta_*) = 0$, $l = 1$ and $E^2 \sim \kappa X$, so the coefficient is $O(X^2)$ and the potential is $\sim d^2 \kappa / X$. The form of $\mathcal{S}_\eta$ then has **infinite negative index** concentrated at $X = 0$. Hence no condition of (NR) type can hold on every leaf: it holds on the midplane (Theorem 9.26) and fails necessarily on the fold leaf.

*Proof.* Elimination and differentiation are done by computer algebra (cv22); $\partial_X H_c = dU_X$ cancels the $\Psi'$ terms. Frobenius at the zero gives (b), and the Hardy constant $\frac{1}{4}$ is sharp. For (c), a dyadic bump at scale $\delta$ gives $\int H_c^2 \Phi'^2 \sim U_X^2 \delta$ against $\int d^2 E^2 l\, \Phi^2 / X^2 \sim \kappa$. The difference is negative for all small $\delta$, and the bumps are mutually disjoint. ▫ **[NUM]** The discretised fold-leaf model has 140, 251, 442, 764 negative eigenvalues at 200–1600 geometric nodes. All near-midplane stagnation points of the six lifts lie in $l < 0$, so $\mathrm{Ri}_* < 0$ there.

**What fails, and what does not.** Everything above is a failure of an **energy method** — of a bounded positive local diagonal multiplier, a Hardy constant or a leafwise symmetrizer. In the parabolic scaling $\partial_\eta \sim \partial_X^2$ away from $\Sigma_*$, the couplings $I_1$–$I_3$ gain one radial derivative, so the principal geometry of Theorems 9.28–9.33 stands there. A height weight $v = \exp(-\kappa \int_0^\eta H_c)$ adds $\frac{1}{2}\kappa H_c^2$ to the budget, at the cost of $O(\kappa)$ radial terms from $\partial_X v$ **[EXACT]**; whether it suffices is **[OPEN]**. The earlier class-level claim that radial diffusion outweighs the fold-leaf negative modes is **withdrawn [FAIL]**: at the axis the pair is not a perturbation (Theorem 9.37).

*The axis fold, scalar part.* Put $r = 2\sqrt{X}$ and $t = \eta - \eta_*$. Then $(X u_X)_X = \Delta_2 u$ and $X^{-1}(X^2 e_X)_X = \Delta_4 e$ exactly (cv24), so $u$ diffuses as a radial function on $\mathbb{R}^2$ and $e$ as one on $\mathbb{R}^4$. Both are Cartesian-regular iff smooth in $r^2$.

**Theorem 9.36 (the scalar axis fold is well posed and smoothing) [EXACT].** Near an axis zero of $H_c$, $H_c = \kappa t + \gamma r^2 + O(t^2, t r^2, r^4)$, with $\kappa = D(1 + \eta_*^2)/d_* + d_* U_\eta(0, \eta_*)$ (the source function of Proposition 9.32) and $\gamma = d_* U_X(0, \eta_*)/4$. Since $H_c(0, \cdot)$ runs from $-D$ to $D$, simple axis zeros alternate in type and **at least one is a source** ($\kappa > 0$). On $Q = \{|x| < R,\ |t| < T\}$ with $|\kappa| T > |\gamma| R^2$, consider $(\kappa t + \gamma |x|^2) q_t - c \Delta q = f$ with $q = g$ on $|x| = R$. (a) At a source fold both $t = \pm T$ are outflow (Fichera), so **no data are imposed on** $t = \text{const}$ **or on the fold**. At a sink fold data are imposed on both. (b) If $c\, j_{0,1}^2 / R^2 > \frac{1}{2}\max(\kappa, 0)$ there is a unique weak solution in $W = \{q \in L_t^2 H_x^1 :\ H_c q_t \in L_t^2 H_x^{-1}\}$ with outflow traces, depending continuously on $(f, g)$. Axis regularity is automatic in $H^1(\mathbb{R}^2)$. (c) If $\gamma \neq 0$ the operator is **hypoelliptic**. With $X_j = \partial_{x_j}$ and $X_0 = H_c \partial_t$, the brackets $[X_j, [X_k, X_0]] = 2\gamma \delta_{jk} \partial_t$ span at the fold point (step three), so $q \in C^\infty$ wherever $f$ is. (d) If $\gamma = 0$ it is not hypoelliptic: $q = \mathbf{1}_{t > 0}$ solves

$\kappa t\, q_t = \Delta q$ in $\mathcal{D}'$. The homogeneous weights are $t \sim r^4$ at the fold point (nilpotent model $\gamma|x|^2\partial_t - c\Delta$), $t \sim r^3$ at transverse zeros (Kolmogorov) and $t \sim r^2$ off $\Sigma_*$. The geometric scaling $t \sim r^2$ of the fold is not a parabolic balance.

*Proof.* (a) The Fichera function on $t = \pm T$ is $\pm H_c$. (b) The energy identity $\iint c|\nabla q|^2 + \frac{1}{2}\int H_c q^2|_{-T}^{T} - \frac{\kappa}{2}\iint q^2 = \iint fq$ holds, with Poincaré on the disc; Lions's theorem on smooth test functions gives existence, and Friedrichs mollification in $t$ gives uniqueness since $H_c$ is Lipschitz (Oleinik–Radkevič). (c) Hörmander's theorem. (d) $t\,\delta(t) = 0$. ▫ **Test A fails.**

*The axis fold, coupled.* At the axis $l = 1$, $E^2 = \frac{\kappa_E}{4}r^2 + O(r^4)$ and $U_X$ is finite. Here $A_X$ is the **disc average** $\mathfrak{M}u = \frac{2}{r^2}\int_0^r u\,\rho\,d\rho$, which is homogeneous of degree 0 with $L^2(r\,dr)$ norm 2 and is **not smoothing**. The pressure is $p = -\frac{\kappa_E}{2}\int_r^R \rho\, e\, d\rho$, so $p_r = \frac{\kappa_E}{2}re$.

**Theorem 9.37 (the coupled axis fold is quasi-elliptic, not ill posed) [EXACT | TRANSCRIPTION].** Assign weights $r \sim \lambda$ and $t \sim \lambda^\beta$. (a) *Supercriticality.* At the scalar-fold weight $\beta = 4$ the product of the pair $d\,l\,\mathfrak{M}\partial_t$ and $d\,\partial_t\mathcal{V}(E^2\cdot)$ has degree $-6$, against $-4$ for the product of the two diffusions. **The pair is supercritical and not relatively bounded by the scalar fold operator.** Exactly, at $l = 1$ the leafwise form is $\langle \mathcal{S}\Phi, \Phi\rangle = \int 2H_c^2\Phi_r^2\,dr/r - \int 2d^2\kappa_E\Phi^2\,dr/r$. On a bump at scale $\delta$ it is negative iff $|H_c| < C_\phi\, d\sqrt{\kappa_E}\,\delta$; on the fold leaf, $H_c \approx \gamma\delta^2$, that holds for all $\delta < C_\phi d\sqrt{\kappa_E}/|\gamma|$. (b) *Principal model.* The pair balances diffusion at $\beta = 3$ with $u \sim \lambda e$. There $H_c\partial_t$, the shear term $dXU_X\mathfrak{M}\partial_t$ and all zero-order terms are subcritical by at least one power of $\lambda$, **so the fold geometry is lower order at the fold point**. The principal model is

$$-d\,\mathfrak{M}(u_t) = 2L\,\Delta_4 e, \qquad d\,p_t = 2L\,\Delta_2 u.$$

(c) *Exact reduction.* Using $\partial_r(r^2\mathfrak{M}v) = 2rv$, $p_r = \frac{\kappa_E}{2}re$ and the intertwining $\Delta_{n+2}\frac{1}{r}\partial_r = \frac{1}{r}\partial_r\Delta_n$, the model is equivalent to

$$u_{tt} = \frac{4L^2}{d^2\kappa_E}(-\Delta_2)^3 u,$$

a **quasi-elliptic operator of type** $(2, 6)$ on radial functions of $x \in \mathbb{R}^2$. With $c_6 := 4L^2/(d^2\kappa_E)$ its symbol satisfies $-\tau^2 - c_6|\xi|^6 \neq 0$ for every $(\tau, \xi) \neq (0, 0)$, being a sum of two non-positive terms that vanish only together, and it is Cartesian-regular at $r = 0$. The sign is sgn$l$: for $l < 0$ the symbol would be $-\tau^2 + c_6|\xi|^6$, vanishing on $\tau = \pm\sqrt{c_6}|\xi|^3$ — dispersive, not elliptic, with no decay across the core and resonances on finite $t$-intervals. The axis fold always has $l = 1 > 0$ by axis regularity, so $l < 0$ is an off-axis comparison only. (d) *Consequence.* The principal coupled fold problem is an **elliptic boundary-value problem in** $(t, x)$. It is ill posed as a march in $t$ (Theorem 9.27). It is well posed with $u$ given at $t = \pm T_{\text{core}}$ — **transmission traces induced by the outer modes, not physical height data** (Theorem 9.39(a)) — together with the radial triple inherited from the original variables at $r = R$:

$$B_1 u = u, \qquad B_2 u = \Delta_2 u = \frac{d}{2L}\,p_t, \qquad B_3 u = \partial_r\Delta_2 u = \frac{d\kappa_E R}{4L}\,e_t,$$

that is, the $u$-trace, the pressure trace (from $d\,p_t = 2L\Delta_2 u$) and the $e$-trace (from $e_t = \frac{4L}{d\kappa_E r}\partial_r\Delta_2 u$). In the global problem the pressure trace is the Volterra integral to $X_a$, so $B_2$ is a **nonlocal** condition, and it is kept as such. The triple is complementing (Lopatinskii). At $\tau = 0$ this follows by factorisation: $w = \Delta u$ satisfies $\Delta^2 w = 0$, $w = \partial_n w = 0$, so $w = 0$, and then $\Delta u = 0$ with $u = 0$. For $\tau \neq 0$, the minimal singular value of the Lopatinskii matrix on $\tau^2 + |\xi'|^6 = 1$ is 0.213 **[NUM]** (cv26). The multiplier $-\Delta_2 v$ gives the coercive form $\iint \nabla u_t \cdot \nabla v_t + c_6 \iint \Delta^2 u\,\Delta^2 v$ on $H_t^1 H_x^1 \cap L_t^2 H_x^4$, so the transmission traces lie in $[H^1, H^4]_{1/2} = H_x^{5/2}$. That form is a **nonlocal, Volterra-adapted symmetrizer**, outside the scope of

> Theorem 9.34(c). The infinite negative index of Theorem 9.35(c) is the indefinite pair mass of this elliptic system, **not an instability**. (e) *Away from the axis* the exact modal structure is Theorem 9.38, and the mode zeros are classified in Theorem 9.39.

*Proof.* (a)–(b) Degree counting, and the change of variables $X = r^2/4$, $dX = r\,dr/2$, $\partial_X = \frac{2}{r}\partial_r$. (c) Computer algebra (cv24). (d) Multiply by $-\Delta_2 v$; the boundary terms vanish under the inherited triple. (e) The high-frequency symbols $A_X \sim 1/(ikX)$ and $\mathcal{V} \sim -1/(ikX)$ in the $2\times 2$ determinant. ▫ **[NUM]** The discretised principal pair on $(0,1)$ has real $\pm$ pairs of rates for $l = +1$ ($|\sigma|$ up to $3.45\cdot 10^9$, imaginary parts zero) and purely imaginary ones for $l = -1$.

**Axis-fold obstruction tests.** - *(A) Scalar fold ill posed:* no (Theorem 9.36). - *(B) Diffusion does not dominate:* **true, but it rules out only the perturbative method.** The pair is supercritical, and the principal model it sets is quasi-elliptic (Theorem 9.37). - *(C) Pressure loss:* no. The pressure gains a radial derivative. Eliminating $e$ costs three, which is exactly the $(2,6)$ balance. - *(D) Volterra instability:* only for marching, since $|\sigma| \sim k^3$ with both signs at $H_c \approx 0$ for $l > 0$. This is already **[FAIL]** as an initial-value problem and harmless for a boundary-value problem. - *(E) No admissible data surface:* no. A source fold needs no data on $t =$ const: **[EXACT]** for the scalar model, **[CLASS]** mode by mode for the coupled one. - *(F) Axis-regularity conflict:* no. $\Delta_2$, $\Delta_4$ and the intertwiner $\frac{1}{r}\partial_r$ preserve smoothness in $r^2$, and $H_c = \kappa t + \gamma|x|^2$ is smooth.

**No obstruction test reaches the PDE.** Regularity is not addressed beyond Theorem 9.36(c). Nothing here gives smoothness of the coupled system through the fold, the sinks, the tangencies or $(X_s, 0)$.

**Obstruction tests.** *(A) Source sign not steerable:* not ruled out — no sink is forced topologically. But sinks occur in every tested lift **[NUM]**, and steering is only through the handover trace. *(B) Tangency topologically forced:* **yes, at the axis** (Theorem 9.33(a)); not in the interior. *(C) Midplane closure forces sinks:* open — generic in the tested lifts, not proved. *(D) Geometry control destroys zero stress:* no — geometry changes are trace changes inside the stress-free class. *(E) Geometry control breaks the handover:* no rank mismatch **[COND]**. *(F) The corner cannot be isolated:* **yes** — the axis fold is a second forced singular point.

**Compatibility verdict.** Take (NR) and the scalar ($\mathrm{NR}_\eta$). Every exact bound found allows them together: Test A forces only $X_a < 8$, and amplitude decouples. In every tested lift they fail together **[NUM]**, but that is not proved. The question is also superseded. The scalar ($\mathrm{NR}_\eta$) is not a coercivity condition of the system (Theorem 9.34), and the operator that joins it to (NR) is necessarily indefinite at the forced axis fold (Theorem 9.35). **Outcome: energy-method failure only, independent of (NR).** The Hardy/Lax–Milgram/Lions route with bounded positive local diagonal multipliers cannot give weak existence for the inboard system, so weak existence is not claimed. Condition (G) is unaffected. The handover trace is data, and subtracting it needs a trace with an $\eta$-derivative — the class in which the layer's correction functions of Theorems 9.13 and 9.6 are smooth **[COND on Theorem 9.6]**. But the layer has $l \equiv -h < 0$, so the handover edge lies where diagonal symmetrization fails.

**Remark (R1, R15).** (NR) and the scalar ($\mathrm{NR}_\eta$) are not two slices of one condition, since each drops the other's term. (NR) is the swirl–pressure pair at $H_c = v$; the scalar estimate keeps $H_c$ and deletes that pair. The exact common object is the height mass operator $\mathfrak{A}_\eta$ of Theorem 9.35, whose midplane slice is (NR). **Merged at the operator, separate as estimates.**

*Microlocal transmission.* Freeze coefficients at $X > 0$ with $kX \gg 1$ (in $r$: $\xi r \gg 1$). Modulo a rank-one operator — the two Volterra base points 0 and $X_a$ — and terms of order $-2$, every off-$H_c$ entry of $\mathfrak{A}_\eta$ carries the factor $\partial_X^{-1}$. The shear entry does so **exactly**, since $dXU_X A_X = H_{c,X}\partial_X^{-1}$ with $\partial_X H_c = dU_X$. Hence $\mathfrak{A}_\eta = H_c I + \partial_X^{-1}\tilde{K}$ with $\tilde{K} = \frac{d}{X}\begin{pmatrix} 0 & -l \\ E^2 & -XU_X \end{pmatrix}$, and the principal symbols are

$$\widehat{\mathfrak{A}} = H_c\, I + \frac{d}{ikX}\begin{pmatrix} 0 & -l \\ E^2 & -XU_X \end{pmatrix}, \qquad \widehat{\mathfrak{D}} = 2LXk^2\, I + ik\,\mathrm{diag}\big(XW - 4Ll,\ XW - 2L\big).$$

**Theorem 9.38 (exact modal structure of the height mass operator) [EXACT | TRANSCRIPTION].** For $l > 0$ put $\varepsilon = dE\sqrt{l}/(kX)$, $\zeta = H_c/\varepsilon$ and $\vartheta = XU_X/(2E\sqrt{l}) = (4\mathrm{Ri})^{-1/2}$. Near the axis $\zeta = \xi H_c/(d\sqrt{\kappa_E})$ and $\vartheta \simeq rU_X/(4\sqrt{\kappa_E})$. (a) The eigenvalues of $\widehat{\mathfrak{A}}$ are $m_\pm = \varepsilon\big(\zeta \pm \sqrt{1-\vartheta^2} + i\vartheta\big)$ and the rates are $\sigma_\pm = -2LXk^2/m_\pm$. At $\vartheta = 0$ these are $-2LXk^2/(H_c \pm \varepsilon)$. (b) The right eigenvectors are $v_\pm = (-l, \mu_\pm)$ with $\mu_\pm = E\sqrt{l}\,(-\vartheta \pm i\sqrt{1-\vartheta^2})$. The left eigenvectors are the rows of $V^{-1} = \frac{1}{l(\mu_+ - \mu_-)}\begin{pmatrix} \mu_- & l \\ -\mu_+ & -l \end{pmatrix}$, and the projectors are $P_\pm = v_\pm w_\pm^{\mathsf{T}}$. (c) **Eigenvectors and projectors are independent of $H_c$, $\zeta$ and $k$.** In the balanced variables $(e/\sqrt{l},\ u/E)$ — the amplitude relation $u \sim re$ of Theorem 9.37(b) — the condition number is exactly $\sqrt{(1+|\vartheta|)/(1-|\vartheta|)}$. (d) Diagonalisation is uniform for $|\vartheta| \le \vartheta_0 < 1$. It fails only at $|\vartheta| = 1$, i.e. $\mathrm{Ri} = \frac{1}{4}$, where $K$ is a Jordan block. **The critical Richardson number of Theorem 9.35(b) is the coalescence point of the two modes.** Near the axis this happens only at $r \simeq 4\sqrt{\kappa_E}/|U_X|$, outside the fold neighbourhood. (e) Hence **the transition $|\zeta| \sim 1$ contains no eigenvector transition**. The quasi-elliptic core ($|\zeta| \ll 1$) and the outer fold ($|\zeta| \gg 1$) are the same two modes, differing only in their scalar masses $m_\pm$.

*Proof.* The characteristic polynomial, eigenvectors and balanced condition numbers were checked by computer algebra and random samples (cv25). The factor $\partial_X^{-1}$ comes from $A_X = X^{-1}\partial_X^{-1}$ and $\mathcal{V} = -\partial_X^{-1}(\cdot/X) +$ rank one. ▫

**Theorem 9.39 (mode zeros and the transmission normal form) [EXACT | TRANSCRIPTION].** (a) *Coarse scale.* $\partial_\eta \mathrm{Re} m_\pm = \partial_\eta H_c \pm \partial_\eta\big(\varepsilon\sqrt{1-\vartheta^2}\big) = \kappa + O\big((kX)^{-1}\big)$. So for $kX \ge k_0 X := C\sup|\partial_\eta(dE\sqrt{l}/X)|/\kappa$, both mode zeros $\zeta = \mp\sqrt{1-\vartheta^2}$ are **sources at a source fold, one per mode**. In the core $\mathrm{Re} m_+ > 0$ carries mode + upward and $\mathrm{Re} m_- < 0$ carries mode − downward. Each core end therefore receives exactly one incoming mode, emitted by that mode's source, which needs no data. (b) *Fine scale.* The source picture is **not** local. The imaginary part $\mathrm{Im} m_\pm = \varepsilon\vartheta = H_{c,X}/(2k)$ and the tilt of the zero across $1/k$, $H_{c,X}/k$, come from the same $U_X$ and are **locked in ratio** $\frac{1}{2}$. A frozen turning-point analysis that keeps one and drops the other is inconsistent; the complex-shifted source $(\kappa\tau + i\delta)^{-\nu}$, which would carry a spurious kernel, is an artefact of that kind. At the Kolmogorov weight $x = X - X_* \sim \lambda$, $\tau \sim \lambda^3$ the tilt, the shear and the pair shift are all critical and $\kappa\tau$ is subcritical. Each mode zero is then, in units of $H_{c,X}$, the scale-invariant normal form

$$\mathcal{N}_{\mathrm{Ri}}: \qquad \big(x + c_*\,\partial_x^{-1}\big)\,a_\tau = \nu\, a_{xx}, \qquad c_* = -\tfrac{1}{2} \pm i\sqrt{\mathrm{Ri} - \tfrac{1}{4}}.$$

(c) *Exact connection.* A Laplace transform in $\tau$ at rate $\sigma$ gives $\sigma\big(xa' + (1+c_*)a\big) = \nu a'''$, solved by $a = \int_\Gamma s^{c_*} e^{xs - \nu s^3/(3\sigma)}\,ds$. **The exponent $c_*$ is the Frobenius exponent of Theorem 9.35(b)**, so the leafwise mass operator and the microlocal normal form are Fourier–Laplace duals. The connection constants are values of $\Gamma(1+c_*)$, and $|\Gamma(\frac{1}{2} + i\omega)|^2 = \pi/\cosh\pi\omega \le \pi$. They are therefore bounded uniformly for $\mathrm{Ri} \ge \frac{1}{4}$ and exponentially small as $\mathrm{Ri} \to \infty$, where the two sides decouple — which is the elliptic core. The scalar normal form has no singularity at $\mathrm{Ri} = \frac{1}{4}$: there $c_* = -\frac{1}{2}$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The logarithm belongs to the leafwise double root and to the Jordan coupling of Theorem 9.38(d). (d) *Uniformity.* $\mathcal{N}_{\mathrm{Ri}}$ is exactly invariant under $x \to \lambda x$, $\tau \to \lambda^3\tau$, so **any estimate for it is uniform in $k$**. On $x \in \mathbb{R}$ with $a = \Psi_x$, $\partial_\tau\|\Psi_{xx}\|^2 = \frac{2}{\nu}\|\Psi_\tau\|^2$ for every $\mathrm{Ri} \ge \frac{1}{4}$, since only $\mathrm{Re} c_* = -\frac{1}{2}$ enters. The estimate itself is Theorem 9.40.

*Proof.* (a) Differentiate $m_\pm$ and read off the signs. (b) The Weyl symbol of $H_c - H_{c,X}\partial_X^{-1}$ is $H_c + iH_{c,X}/k$; the rest is degree counting. (c) Substitution (cv25). (d) Multiply by $\bar{\Psi}_\tau$ and take real parts. ▫

**Frequency closure and derivative count.** - *High frequency* ($\xi r \gg 1$): Theorems 9.38–9.40. - *Axis scale* ( $\xi r \lesssim 1$): with $|\zeta| \ll 1$ the exact model is Theorem 9.37; with $|\zeta| \gg 1$ it is Theorem 9.36. - *The corner* $\xi r \sim 1$, $|\zeta| \sim 1$, i.e. $|t| \sim d\sqrt{\kappa_E}\, r/\kappa$, is homogeneous under no single scaling and is **[OPEN]**. - *Low frequency* ($\xi \lesssim 1/R$ ): finite-dimensional. A high-frequency estimate plus this compact part gives a Fredholm operator, so well-posedness is equivalent to a finite non-resonance condition, of the same species as Theorem 9.26. - *Derivatives:* $e \to u$ costs three radial derivatives per $\partial_t$, consistent with the weight $t \sim r^3$. Pressure reconstruction gains one. The projectors are constant in balanced variables. The crossing is scale-invariant, so no loss grows with $k$; outer modes gain $e^{-c\xi^2/\kappa}$ across the transition width $1/(\kappa\xi)$ **[CLASS]**. **No derivative loss at the principal level.**

**Transmission obstruction tests.** - *(A) Loss of diagonalisability:* locally no ($\vartheta \to 0$ at the axis). It occurs only at $\mathrm{Ri} = \frac{1}{4}$, off the fold, and rules out the modal method there only. - *(B) Bad mode surface:* no. The zeros are sources at coarse scale and $\mathcal{N}_{\mathrm{Ri}}$ at fine scale; neither is a sink or needs height data. - *(C) Transmission constant blows up:* no. The estimate of Theorem 9.40 has constant 1, uniformly in $k$ and Ri. - *(D) Trace mismatch:* no. There is exactly one incoming mode per core end. - *(E) Radial boundary mismatch:* no. The triple is inherited and complementing. - *(F) Derivative loss:* no at the principal level. - *(G) Low-frequency obstruction:* **not excluded.** Fredholm theory allows finitely many resonances; this is condition (T3).

No obstruction test reaches the fold.

**Remark (R1, R15).** The hypoelliptic outer fold and the quasi-elliptic core have the same eigenvectors (Theorem 9.38(c)). They reconstruct each other through the scalar mode masses $m_{\pm}(\zeta)$, and the transmission exponent is the leafwise Frobenius exponent (Theorem 9.39(c)). **Merged: they are two resolution regimes of one boundary-value problem.**

*Closing (T1).* Take $\partial_x^{-1}$ to be the Fourier multiplier $1/(ik)$, the symmetric base point; other base points differ by a rank-one operator supported at $k = 0$. Write $|D|^c$ for the multiplier $|k|^c$.

> **Theorem 9.40 (the fold normal form is the Kolmogorov equation in a nonlocal gauge) [EXACT | TRANSCRIPTION].** Let $c = -\frac{1}{2} \pm i\omega$, $\omega = \sqrt{\mathrm{Ri} - \frac{1}{4}} \geq 0$. (a) *Conjugation.* $(x + c\,\partial_x^{-1})\,|D|^c = |D|^c\, x$. Hence $a = |D|^c b$ solves $\mathcal{N}_{\mathrm{Ri}}$ iff $b$ solves
>
> $$x\, b_\tau = \nu\, b_{xx},$$
>
> the transverse-zero (Kolmogorov) model of the scalar theory. Equivalently, $\widehat{\Psi} = k^{c-1}\varphi$ turns $\mathcal{N}_{\mathrm{Ri}}$ into $\varphi_{k\tau} = i\nu k^2 \varphi$, in which $c$ does not appear. **At fine scale the Richardson number is a gauge.** It acts only through the unitary $|D|^{\pm i\omega}$ and the fixed half-derivative $|D|^{-1/2}$. (b) *Nonlocal symmetrizer.* Since $2\mathrm{Re}c + 1 = 0$, the multiplier $|D|\bar{a}$ gives, for $\mathcal{N}_{\mathrm{Ri}}$ with source $f$,
>
> $$\frac{1}{2}\partial_\tau \int x\,|b|^2 dx + \nu\,\|a\|^2_{\dot{H}^{3/2}} = \mathrm{Re}\langle f, |D|a\rangle, \qquad b = |D|^{-c}a,$$
>
> and $\|a\|_{\dot{H}^{s+1/2}} = \|b\|_{\dot{H}^{s}}$ for every $s$ and every real $\omega$. (c) *Well-posedness.* Prescribe the incoming half-range traces $b(\cdot, 0)$ on $x > 0$ and $b(\cdot, T)$ on $x < 0$, i.e. of $|D|^{-c}a$. Then $\mathcal{N}_{\mathrm{Ri}}$ on $\mathbb{R} \times (0, T)$ has a unique solution with $a \in L^2_\tau \dot{H}^{3/2}_x$ and weighted traces. For $f = 0$,
>
> $$\nu\|a\|^2_{L^2\dot{H}^{3/2}} + \frac{1}{2}\int_{x>0} x|b(T)|^2 + \frac{1}{2}\int_{x<0} |x|\,|b(0)|^2 = \frac{1}{2}\int_{x>0} x|b(0)|^2 + \frac{1}{2}\int_{x<0} |x|\,|b(T)|^2.$$
>
> The constant is 1: uniform in $k$ by scale invariance and uniform in $\mathrm{Ri} \in [\frac{1}{4}, \infty)$. **No data are imposed at the zero** $x = 0$, and one incoming trace enters at each end, as counted in Theorem 9.39(a). (d) *Localisation.* For a smooth cut-off $\chi$, $[\chi, |D|^{-c}]$ has order $-\frac{1}{2}$ and its low-frequency part is smoothing. That commutator is compact **outside the mass**. Inside the mass it creates a skew term that no energy estimate tolerates (Lemma 9.43), so (c) localises by energy only as an exact whole-line statement.

*Proof.* (a) Under the Fourier transform $x \mapsto i\partial_k$ and $\partial_x^{-1} \mapsto 1/(ik)$, so $\widehat{(x + c\partial_x^{-1})h} = i(\partial_k - c/k)\widehat{h} = i|k|^c \partial_k \big( |k|^{-c} \widehat{h} \big)$, on both half-lines of $k$. The $\varphi$-form is the same computation with $\widehat{\Psi} = \widehat{a}/(ik)$. (b) $\langle (x + c\partial^{-1}) a_\tau, |D| a \rangle = \langle x\, b_\tau, |D|^{\bar{c}+1+c} b \rangle = \langle x\, b_\tau, b \rangle$, and $\langle a_{xx}, |D| a \rangle = -\|a\|^2_{\dot{H}^{3/2}}$. (c) For the Kolmogorov problem on windows $|x| < L$ with Dirichlet ends, the form $\nu \iint \phi_x^2$ plus the outflow terms is coercive on test functions, so Lions's theorem gives existence (Baouendi–Grisvard). The identity in (b) is uniform in $L$, so weak limits exist as $L \to \infty$. Uniqueness follows from the identity after mollifying in $\tau$, which commutes with the $\tau$-independent coefficients. The isometry of (b) transfers everything to $a$. (d) Standard symbol calculus; $|k|^c$ is smooth away from $k = 0$. ▫

**[NUM]** (cv27) The conjugation, the symmetrizer and the isometry hold to relative error $\le 2 \cdot 10^{-13}$ for $\omega = 0,\ 0.7,\ 3,\ 12$. (cv28) The space-time half-range Kolmogorov system is square and nonsingular with **no condition at** $x = 0$. The point value converges ($b(0, T/2) = 0.7152, 0.7125, 0.7112$), the maximum principle holds, and the energy identity balances to 0.3%.

*Obstruction tests on (T1).* - *Ri-dependent resonance:* no — Ri is a gauge. - *Data required at the zero:* no — the energy identity and the square discrete system. - *Conjugated data are not the physical data:* only a localisation effect, compact outside the mass; inside the mass see Lemma 9.43. - *Derivative loss:* no — $a$ is $\frac{1}{2}$ derivative smoother than $b$, exactly. - *Singularity at* $\mathrm{Ri} = \frac{1}{4}$*:* none in the scalar normal form.

**Remark (R15).** The coupled mode zero and the scalar transverse zero of Theorems 9.28–9.31 are **one object in two gauges**. The Richardson number is physical only at $k = 0$ (the Volterra base point, rank one) and through the modal coalescence at $\mathrm{Ri} = \frac{1}{4}$ (Theorem 9.38(d)), which lies off the fold.

*Attacking (T2).* First, a correction to the setting. The corner $\xi r \sim 1$, $|\zeta| \sim 1$ is **not** self-similar under $r \to \lambda r$, $t \to \lambda t$: at $t \sim r$ radial diffusion dominates the mass terms by $\lambda^{-2}$. Its own scale is $t \sim r^3$, over which $H_c \approx h_0 \sim d\sqrt{\kappa_E}\, r$ is frozen. The remark that (T2) is a Mellin problem is **withdrawn [FAIL]**. The corner nevertheless needs no separate treatment, because at the axis the symmetrization that Theorem 9.34 excluded in general becomes available.

> **Theorem 9.41 (Hermitian structure of the coupled fold; (T2) and (T3) closed for the symmetric fold) [EXACT | TRANSCRIPTION].** On the fold neighbourhood $l > 0$ ($l = 1$ at the axis). Take multipliers $\mu e$ and $\nu u$ with $\nu$ constant and $\mu = \nu E^2/l$. This is Theorem 9.34's condition, admissible here because $l$ does not change sign and the weight may degenerate like $E^2 \sim \kappa_E X$; Theorem 9.34 excluded only bounded positive weights across $l = 0$. In $r$ these are the natural measures $\frac{\kappa_E}{4} r^3 dr$ for $e$ (the $\mathbb{R}^4$ measure of $\Delta_4$) and $r\, dr$ for $u$. (a) *Self-adjointness.* Let $\mathcal{A}_{\mathrm{sym}}(t) = H_c\, I + \mathcal{P}$ with $\mathcal{P}(e, u) = (-dl\, A_X u,\ d\, p)$. It is self-adjoint in $\langle Z, Z' \rangle_H = \int (\mu e \bar{e}' + \nu u \bar{u}')\, dX$. Its pair part is an exact derivative: $\mathrm{Re}\langle \mathcal{P} Z_t, Z \rangle_H = -\nu\, \partial_t \mathrm{Re} \int dE^2\, \bar{e}\, A_X u\, dX$ up to zero-order terms from $\partial_t(dE^2)$. (b) *Energy.* On a window $\{r < R\}$ with the lateral data subtracted,
>
> $$\tfrac{1}{2}\partial_t \langle \mathcal{A}_{\mathrm{sym}} Z, Z \rangle_H + \|Z\|_V^2 - \tfrac{\kappa}{2}\|Z\|_H^2 + \mathcal{R}(Z, Z) = \mathrm{Re}\langle F, Z \rangle_H,$$
>
> where $\|Z\|_V^2 = 2L \big( \int \frac{\kappa_E}{4} r^3 |e_r|^2 dr + \int r |u_r|^2 dr \big)$ at leading order. $\mathcal{R}$ collects the zero-order couplings, the radial drift and $\partial_t$ of the weights, and satisfies $|\mathcal{R}| \le C(R)\|Z\|_V^2$ with $C(R) \to 0$. (c) *Pair bound.* $|\langle \mathcal{P} Z, Z \rangle_H| \le \frac{d\sqrt{\kappa_E} R}{2} \|Z\|_H^2$ at leading order, from $|\Psi| \le r \|u\|_r / \sqrt{2}$ and AM–GM. (d) *Well-posedness.* Suppose $2L j_{0,1}^2 / R^2 > \frac{\kappa}{2} + C(R)$ and, at a source fold, $\kappa T > |\gamma| R^2 + \frac{d\sqrt{\kappa_E} R}{2}$. Then $\mathcal{A}_{\mathrm{sym}}(\pm T)$ is definite and both height ends are outflow. The symmetric fold problem — pair, $H_c$, diffusion and every lower-order term except the shear — has a unique weak solution in $\{Z \in L^2 V :\ \mathcal{A}_{\mathrm{sym}} Z_t \in L^2 V'\}$, depending continuously on the data. **No data are imposed on** $t = \pm T$ **and there is no resonance.** (e) One energy covers the core ($|\zeta| \ll 1$), the corner and the outer fold ($|\zeta| \gg 1$) at once. **(T2) and (T3) are closed for the symmetric fold, and the corner was an artefact of microlocal splitting.**

*Proof.* (a) Pair the $e$-row with $\nu E^2 \bar{e}/l$ and the $u$-row with $\nu\bar{u}$. Fubini turns $\int \bar{u}\, p_t$ into $-\int E^2 e_t A_X \bar{u}$, so the two pair terms add to a real $t$-derivative. (b) Integrate by parts in $r$ and apply the Poincaré constants $j_{0,1}^2/R^2$ for $\Delta_2$ and $j_{1,1}^2/R^2$ for $\Delta_4$. Zero-order terms are $O(R^2)\|Z\|_V^2$ and drift terms $O(R)\|Z\|_V^2$. (c) Cauchy–Schwarz for $\Psi = \int_0^r u\rho\, d\rho$, then $\frac{\kappa_E}{4}\|e\|^2 + \|u\|^2 \geq \sqrt{\kappa_E}\|e\|\|u\|$. (d) Lions's theorem on test functions vanishing at the inflow parts, of which there are none. Uniqueness follows from (b) after Friedrichs mollification in $t$, since $\partial_t \mathcal{A}_{\mathrm{sym}}$ is bounded. ▫ **[NUM]** (cv29) The discrete pair is Hermitian to $1.5 \cdot 10^{-16}$. Its form lies in $[-0.374, 0.374]\|Z\|_H^2$ against the bound 0.45. A space-time solve with **no height data** is nonsingular and converges ($u(0.5, 0) = 0.01249, 0.01214, 0.01197$), with outflow traces of order $10^{-6}$.

*Obstruction tests on (T2).* All fail for the symmetric fold, and the fifth confines what remains to Theorem 9.40's regime. - *The corner needs extra data:* no; both ends are definite and outflow. - *A corner resonance:* no; the form is coercive on small windows. - *The degenerate weight loses axis regularity:* no; $\frac{\kappa_E}{4} r^3 dr$ is the $\mathbb{R}^4$ measure in which $e$ is Cartesian-regular. - *Interior degeneracy of $\mathcal{A}_{\mathrm{sym}}(t)$:* irrelevant, since only the ends enter the data count. - *The shear:* see below.

> **Proposition 9.42 (the residue is the shear's skew mass) [EXACT | TRANSCRIPTION].** (a) With the weights of Theorem 9.41, the only principal term that is not self-adjoint is the shear $-dXU_X A_X(u_t)$ in the $u$-row. For locally constant $U_X$ its symmetric part has rank one, and its skew part is the Volterra kernel
>
> $$S\,u = -\frac{dU_X}{2}\int \mathrm{sgn}(X - x)\, u(x)\, dx, \qquad \|S\|_{\mathrm{HS}} \leq \frac{|\gamma| R^2}{2}.$$
>
> $S$ is small on small windows but acts on $u_t$, which the energy does not control; that is the cos$k\eta$ test of Theorem 9.34(c). **[NUM]** The discrete shear block is 99.5% skew. (b) *No bounded correction of the weights removes $S$.* The first-order Sylvester equation for a correction $G_1$, pair-free part $H_{c,X}[G_1, X] = -2\nu S$, has the exact kernel $g(X, x) = -\nu/|X - x|$, using $H_{c,X} = dU_X$. That kernel diverges logarithmically (symbol $\sim\log|k|$). This is the exact reason the fine-scale symmetrizer of Theorem 9.40 had to be of order one, $|D|$, and not a weight. (c) *Frequency-split symmetrizer* — **[FAIL] by Lemma 9.43**. Relative to the pair the shear has size $\vartheta \simeq rU_X/(4\sqrt{\kappa_E}) \to 0$ at the axis, which suggested gluing the Hermitian symmetrizer ($\xi r \leq C_0$, correction cost $\vartheta \log C_0$) to the gauge of Theorem 9.40 ($\xi r \geq C_0$, errors $O(1/C_0)$). But the gluing errors are skew **mass** terms. However small, they destroy coercivity for a degenerate mass.

*Obstruction test on the shear.* Can the skew mass create an instability? Only where the symmetric mass does not dominate it, i.e. at the mode surfaces at high frequency. The test fails at PDE level (Proposition 9.44 and the numerics below), but not through (c). **[NUM]** (cv29) With the shear on, the no-data space-time solve stays nonsingular and convergent ($u(0.5, 0) = 0.01324, 0.01285, 0.01264$, about 5% from the symmetric fold), at the height frequencies the grid resolves.

*Building (S).*

> **Lemma 9.43 (degenerate masses need exact symmetrizers) [EXACT].** Let $G(t)$ be any multiplier, local or nonlocal and possibly unbounded. Suppose $B(\phi, \phi) = \mathrm{Re}\iint \langle \mathcal{A}\phi_t + \mathcal{L}\phi,\ G\phi\rangle$ is bounded below by $c\|\phi\|_{\mathcal{N}}^2$ on smooth test functions, for a norm $\mathcal{N}$ with no $t$-derivatives. Then $K(t) := \frac{1}{2}\left(G^*\mathcal{A} - \mathcal{A}^* G\right)(t) \equiv 0$. For $t$-independent $G$ and $\mathcal{A}(t) = \kappa t + \mathcal{A}_0$ this means $G = G^*$ and $G\mathcal{A}_0 = \mathcal{A}_0^* G$ **exactly**. Consequently: - (i) no approximate symmetrizer works, however small or compact its error, so Proposition 9.42(c) fails; - (ii) cut-off localisation of an exact symmetrizer is forbidden inside the mass (Theorem 9.40(d), corrected); - (iii) (S) is equivalent to an exact $G$ on the window with $\mathrm{Sym}(G\mathcal{L})$ coercive.

*Proof.* Take $\phi = \chi(t)\left(\phi_1 \cos kt + \phi_2 \sin kt\right)$ with fixed radial profiles. The Hermitian part of $G^* \mathcal{A}$ gives $\frac{1}{2}\partial_t$ plus terms bounded in $k$. The skew part gives $k\,\langle K\phi_2, \phi_1\rangle \int \chi^2 + O(1)$, while $\|\phi\|_{\mathcal{N}} = O(1)$. Choosing the sign makes $B$ unbounded below unless $K \equiv 0$. ▫ This is why a degenerate mass differs from a uniformly positive one: where the mass is invertible, $Z_t$ is controlled by the equation and lower-order skew mass terms are harmless.

**[NUM] The exact-symmetrizer landscape** (cv30–cv35, Hermitian coordinates, window $R = 1$): - With shear the discrete mass $\mathcal{A}_0 = \gamma r^2 +$ pair $+$ shear has **exactly real spectrum** at every resolution. Its eigenvector condition number grows like $N^{1/2}$ (8.5, 12.3, 17.2, 23.0), the same rate as the proven normal form of Theorem 9.40 on a window (9.6, 13.5, 19.1). In both, the similarity to a self-adjoint operator has order ½. - The optimal exact symmetrizer $V^{-*} D V^{-1}$, maximised over $D \geq 0$ by supergradient ascent, has coercivity margin $-0.025$, $-0.384$, $-1.03$ for the fold. **For the proven control it is also negative** ($-0.056$, $-0.110$, $-0.303$); both shear-free cases give $+1$. - The Laplacian-power family $\mathrm{diag}(L_4^{s/2}, L_2^{s/2})$, scanned by continuum-convergent low-mode compressions, has no exact order for the fold (minimal skew ratio 0.19). **It has none for the control with $c = -\frac{1}{2}$ either** (minimum 0.05), although $|D|$ is exact there on $\mathbb{R}$. - *Verdict:* these searches are **inconclusive about (S)**, and conclusive that exact symmetry is a whole-line property destroyed by windows.

So the energy route on windows is blocked, and blocked also where the PDE is proved well posed. The next test must bypass multipliers.

> **Proposition 9.44 (the fold is invertible in height frequency without a multiplier) [EXACT for the symmetric fold and for $|\tau| \leq \lambda_1/(2\|S\|_H)$; NUM beyond].** For the $t$-invariant fold operator $\mathcal{A}_0\partial_t + \mathcal{L}$ on a window,
>
> $$\mathrm{Re}\langle(\mathcal{L} + i\tau\mathcal{A}_0)z, z\rangle_H = \langle \mathcal{L}z, z\rangle_H - \tau \mathrm{Im}\langle Sz, z\rangle_H,$$
>
> since the Hermitian part of $\mathcal{A}_0$ contributes only imaginary values. Hence $\|(\mathcal{L} + i\tau\mathcal{A}_0)z\|_H \geq \lambda_1 \|z\|_H$ for every $\tau$ in the symmetric fold, and $\geq \frac{\lambda_1}{2}\|z\|_H$ for $|\tau| \leq \lambda_1/(2\|S\|_H)$ with shear. **[NUM]** (cv36) Over $\tau \in \pm[10^{-2}, 10^5]$, $\min_\tau \sigma_{\min}(\mathcal{L} + i\tau\mathcal{A}_0)/\lambda_1 = 1.0000$, attained at $\tau = 0$, with shear on and off at $N_r = 50, 100, 200$, and for the control on the grid. So as far as tested, the $t$-invariant fold on $\mathbb{R}_t$ satisfies $\|Z\|_{L^2 H} \leq \lambda_1^{-1}\|F\|_{L^2 H}$ by Plancherel. **The obstruction on windows is a method obstruction, not a PDE obstruction.**

*Proof.* Take real parts and apply Poincaré on the window; $|\mathrm{Im}\langle Sz, z\rangle| \leq \|S\|\|z\|^2$. ▫

*The large-height-frequency resolvent, by quasimode exclusion.* This is proved at operator level: the object is $\mathcal{R}(\tau) = \mathcal{L} + i\tau\mathcal{A}_0$, not a quadratic form.

**Target.** There exist $\tau_0, c_0 > 0$ with $\|\mathcal{R}(\tau)z\|_H \geq c_0\|z\|_H$ for $|\tau| \geq \tau_0$ and $z$ in the fold domain with the inherited radial conditions. Theorem 9.60 proves this target. An all-frequency lower bound additionally requires non-resonance on the compact frequency interval between this result and the small-frequency range of Proposition 9.44.

> **Lemma 9.45 (frequency confinement) [EXACT | TRANSCRIPTION].** In the Hermitian coordinates of Theorem 9.41, $\mathrm{Im}\langle Sz, z\rangle_H = -\nu\, dU_X \mathrm{Im}\langle \partial_0^{-1} u, u\rangle$: the rank-one base-point part of $S$ is real and drops out. For $s < \frac{1}{2}$, $\partial_0^{-1}: H^{-s} \to H^{1-s}$, because $\mathbf{1}_{[0,X]} \in H^s$. With $H^{-s} = D(\mathcal{L}^{-s/2})$ (no boundary compatibility below $\frac{1}{2}$),
>
> $$\mathrm{Re}\langle \mathcal{R}(\tau)z, z\rangle_H \;\geq\; \langle \mathcal{L}z, z\rangle_H - C_s\,|\tau|\,\|\mathcal{L}^{-s/2}z\|_H^2.$$
>
> Hence a normalised $\varepsilon$-quasimode has spectral measure $\mu$ of $\mathcal{L}$ with $\int \lambda\, d\mu \leq \varepsilon + C_s|\tau| \int \lambda^{-s} d\mu$, and in particular $\mu(\lambda \geq \Lambda) \leq (\varepsilon + C_s\lambda_1^{-s}|\tau|)/\Lambda$. The high side is thus a theorem: quasimodes cannot live at $\lambda \gg |\tau|$. The sharp whole-line gain would be $s = \frac{1}{2}$ (radial frequency $\xi \lesssim |\tau|^{1/3}$); at the endpoint $s < \frac{1}{2}$ is

> what the window allows. The real part alone cannot exclude the low side, because cross-frequency terms of $i\tau\mathcal{A}_0$ do not decouple.

*Contradiction setup.* Suppose $|\tau_n| \to \infty$, $\|z_n\|_H = 1$ and $\|\mathcal{R}(\tau_n)z_n\|_H \to 0$. By Lemma 9.45 the $z_n$ are confined to $\lambda \lesssim |\tau_n|$. A quasimode needs both $\nu\xi^2 \lesssim \tau\,|\text{skew}|$ and $\|\mathcal{A}_0 z\| \lesssim \xi^2/|\tau|$, so it must sit near the near-kernel of the mass symbol $m_\pm(r,\xi) = \gamma r^2 \pm b/\xi + i$ (shear), $b = d\sqrt{\kappa_E}$ (Theorem 9.38, in $r$-units).

> **Proposition 9.46 (local model list; global exhaustion conditional) [EXACT | TRANSCRIPTION for the local models and their specified kernels; COND for exhaustion; CLASS for the exponents].** The local models proposed for large-$\tau$ blow-ups are the following. Their kernel statements do not by themselves show that every window quasimode transfers to one of these domains. The global transport and pressure-trace issue is treated below, after Lemma 9.50. - (i) *Regular point* ($m_\pm(r_0,\xi) \neq 0$). The frozen symbol $\nu\xi^2 + i\tau m_\pm$ is nonzero: its real part is $\geq 0$ and vanishes only at $\xi = 0$, where $b/\xi$ blows up. The operator is invertible. - (ii) *Mode zero* ($\gamma r_0^2 = \mp b/\xi_0, r_0 > 0$). After blow-up at the Kolmogorov scale the operator is $\mathcal{N}_{\mathrm{Ri}}$ with $\mathrm{Ri} = 4\kappa_E/(r_0^2 U_X^2)$. **Only after** passing to the whole-line limit, the gauge of Theorem 9.40 conjugates it to the complex Airy operator $-\nu b'' + i\tau' x\, b$. The Airy equation has no nonzero tempered kernel. Transfer through the singular gauge requires a domain: $|D|^{-c}$ cannot simply be applied to arbitrary tempered distributions at zero frequency. Lemma 9.50 supplies the stronger uniform lower bound on the gauge-compatible physical graph domain, also as $\mathrm{Ri} \to \infty$, because $\omega$ enters through a unitary factor. The shear appears here only as $\mathrm{Re}\, c = -\frac{1}{2}$ and is removed exactly (Test E fails). - (iii) *Axis core* ($r_0 = 0$, radial scale $\rho$ and frequencies $|\xi| \asymp \rho^{-1}$). Here $b\rho \gg \gamma\rho^2$ and the shear is $O(\rho)$ relative, so the blow-up operator is the symmetric pair-plus-diffusion core on whole-space radial functions. $\mathrm{Re}\langle\tilde{\mathcal{R}}z, z\rangle = \langle\tilde{\mathcal{L}}z, z\rangle$ gives $z = 0$; equivalently $(\tau^2 + c_6(-\Delta_2)^3)u = 0$ gives $u = 0$ (Theorem 9.37). - (iv) *Boundary* $r = R$. A mode zero lies at $r_*(\xi) = \sqrt{b/(\gamma\xi)} \to 0$ as $\xi \to \infty$, so none reaches $r = R$ at blow-up frequencies. The frozen mass is $\gamma R^2 \neq 0$. On the half-line, $\mathrm{Im}\langle(-\nu\partial_y^2 + i\tau m_0)z, z\rangle = \tau m_0\|z\|^2$ gives $z = 0$. The pressure base point is rank one and absent from this local principal symbol. Its global matching effect cannot be discarded on that basis: Proposition 9.52 identifies an inviscid reflection relation that it carries. The complementing local boundary calculation does not by itself transfer this relation through a viscous boundary layer.
>
> *Scales.* Without the pair, the double zero of $\gamma r^2$ at the axis would give the complex-oscillator scale $\xi \sim |\tau|^{1/4}$ — the origin of the $\frac{1}{4}$ in the delicate band. At that scale the pair ($\sim b\,\tau^{-1/4}$) beats $\gamma r^2$ ($\sim\tau^{-1/2}$), so that candidate dies. The band $|\tau|^{1/4} \lesssim \xi \lesssim |\tau|^{1/3}$ instead hosts mode-zero packets. Optimising $\nu(\xi_0^2 + w^{-2}) + \tau(2\gamma r_0 w + b\,\xi_0^{-2} w^{-1})$ along $\gamma r_0^2 = b/\xi_0$ gives the constructed family $\xi_0 \sim \tau^{4/13}$, $w \sim \tau^{-3/13}$, $r_0 \sim \tau^{-2/13}$, costing $\sim\tau^{8/13}$, with $\xi_0 r_0 \sim \tau^{2/13} \to \infty$ so frozen symbols are valid. The axis core costs $\sim\tau^{2/3}$. These are upper bounds from constructions, not the sharp exponent.

*Proof of the kernel statements.* (i) Symbol algebra. (ii) Theorem 9.40(a) on $\mathbb{R}$ and the asymptotics of $\mathrm{Ai}(e^{\pm i\pi/6}x)$. (iii) Theorem 9.41(a) at $\gamma = 0$ and $S = 0$, with Theorem 9.37(c). (iv) The imaginary part on the half-line with Dirichlet data. ▫

**[NUM]** (cv37) $\sigma_{\min}(\mathcal{L} + i\tau\mathcal{A}_0)$ on the fold window:

| $\tau$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|
| shear off | 25.8 | 96.4 | 318.8 | 1153 | 4267 |
| shear on | 27.6 | 73.0 | 236.1 | 842.5 | 3068 |

The values are identical at $N_r = 200$ and 400. The local log-log exponent is $0.554 \to 0.564$ (shear off) and $0.545 \to 0.557$ (shear on), still rising, and the minimising vector is interior ($\langle r\rangle$ from 0.66 to 0.26). The complex-Airy control reproduces 0.666. The resolvent bound therefore holds with **growing** margin, and the

measured exponent sits below the constructed 8/13: some quasimode family is cheaper than the one constructed, so the exponent bookkeeping is not sharp. Only a lower bound matters for the theorem.

*Obstruction tests (large τ).* - *(A) Intermediate-band quasimode:* the local estimates exclude packets whose transfer to those model domains is justified. Theorem 9.60 now exhausts the full window sequence, retaining the outer pressure relation. - *(B) Mass near-kernel:* **succeeds for $\mathcal{A}_0$ alone** — it has essential spectrum at 0 through the mode zeros. Growing resolvent cost is supported by the numerical results and candidate balances; a uniform lower bound for all states has not yet been proved. - *(C) Boundary layer:* fails, by (iv) and the interior minimisers. - *(D) Localisation error:* commutators are $O((\xi r)^{-1}) \sim \tau^{-2/13}$ at the mode-zero scale, and the Volterra base points are rank one. Fails at the level of scaling. - *(E) Shear survives the limit:* fails; it is Re$c = -\frac{1}{2}$, removed exactly by the gauge after blow-up. - *(F) Loss of compactness:* **excluded by Theorem 9.60**; the earlier local model list did not close this gap.

**Remark (R15).** Outer fold, transition, core and shear-corrected fold are one operator seen at different resolutions, and no new blow-up operator appears. Energies are not glued; the kernel of the single operator is excluded.

*The compactness step.* Throughout, inverses are glued, not representations: every limit is taken for the single operator, and gauges are applied only after blow-up.

> **Theorem 9.47 (exact mode decoupling at the axis) [EXACT | TRANSCRIPTION].** Let $D_1 = \frac{1}{r}\partial_r$, mapping radial functions on $\mathbb{R}^2$ to radial functions on $\mathbb{R}^4$, with measures $r\,dr$ and $r^3 dr$. - (a) $\Delta_4 D_1 = D_1 \Delta_2$, $D_1^* e = -\frac{1}{r}(r^2 e)_r$ (boundary term $[r^2 fg]$), and $D_1^* D_1 = -\Delta_2$. - (b) $\mathfrak{M}u = 2D_1 \Delta_2^{-1} u$, since $\int_0^r \rho\, \Delta_2 w\, d\rho = r w_r$; and $\Delta_2 p = -\frac{\kappa_E}{2} D_1^* e$, so $p = -\frac{\kappa_E}{2}\Delta_2^{-1} D_1^* e + \text{const}$. - (c) $U := D_1(-\Delta_2)^{-1/2}$ is an isometry with $U^* \Delta_4 U = \Delta_2$. In the Hermitian coordinates $\tilde{e} = \frac{\sqrt{\kappa_E}}{2} e$, $\tilde{u} = u$ the pair is $d\sqrt{\kappa_E}\begin{pmatrix} 0 & U(-\Delta_2)^{-1/2} \\ (-\Delta_2)^{-1/2} U^* & 0 \end{pmatrix}$. - (d) Hence, for the modes $a_\pm = (\tilde{u} \pm U^* \tilde{e})/\sqrt{2}$:
>
> $$\text{pair} = \pm d\sqrt{\kappa_E}\,(-\Delta_2)^{-1/2}, \qquad \text{diffusion} = -2L\Delta_2\, I \qquad \textbf{exactly}$$
>
> on whole-space radial functions; on a window only boundary defects remain. The mode couplings are $\gamma r^2$ (through $U^* r^2 U - r^2$, of lower order) and the shear. So Theorem 9.38's frozen modal picture is an identity at the axis, and "no eigenvector transition" holds without symbol calculus.

*Proof.* Computer algebra (cv38) for (a) and (b). For (c), $\|Uv\|^2 = \langle D_1^* D_1 (-\Delta)^{-1/2} v, (-\Delta)^{-1/2} v\rangle = \|v\|^2$ and $U^* \Delta_4 U = (-\Delta)^{-1/2} D_1^* D_1 \Delta_2 (-\Delta)^{-1/2}$. ▫

> **Lemma 9.48 (the fold mass has no kernel; its near-kernel is the mode-zero curve) [EXACT | TRANSCRIPTION].** Write $\mathcal{A}_0 z = 0$ with the full shear. The $u$-row gives $\gamma r^2 u_r + d\, p_r = 0$, so $e = -2\gamma r u_r/(d\kappa_E)$. Substituting into the $e$-row with $\Psi = \int_0^r u\rho\, d\rho$ gives
>
> $$\Psi'' - \frac{\Psi'}{r} + \frac{\beta^2}{r^4}\Psi = 0, \qquad \beta = \frac{d\sqrt{\kappa_E}}{\gamma},$$
>
> solved exactly by $\Psi = (r/\beta)\left(aJ_1 + bY_1\right)(\beta/r)$. As $r \to 0$ every nonzero solution oscillates with amplitude $\Psi \sim r^{3/2}$, so $u = \Psi'/r \sim r^{-3/2}$ and $\int |u|^2 r\, dr = \infty$. Hence $\ker \mathcal{A}_0 = \{0\}$ in $H$. The near-kernel oscillates with phase $\beta/r$, i.e. local frequency $\beta/r^2$. That is exactly the mode-zero curve $\gamma r^2 |\xi| = d\sqrt{\kappa_E}$ of Theorem 9.38, which generates $0 \in \sigma_{\text{ess}}(\mathcal{A}_0)$ at the axis. **[NUM]** (cv38) The smallest singular values of the discrete mass fall under refinement ($1.0 \cdot 10^{-5} \to 4.9 \cdot 10^{-7}$) and their vectors move to the axis. A Bessel-phase fit is aliased, since $\beta/r^2$ exceeds the grid's resolvable frequency, and is not used.

**Proposition 9.49 (compactness reduction for the large-frequency resolvent).** Let $|\tau_n| \to \infty$, $\|z_n\|_H = 1$, $\|\mathcal{R}(\tau_n) z_n\|_H \to 0$. - (a) **[EXACT]** $\sup_n \langle \mathcal{L} z_n, z_n \rangle = \infty$. Otherwise $V \Subset H$ gives $z_n \to z$ strongly with $\|z\| = 1$, and for $\phi \in D(\mathcal{L})$, $\langle \mathcal{A}_0 z_n, \phi \rangle = (\langle \mathcal{R} z_n, \phi \rangle - \langle z_n, \mathcal{L}\phi \rangle)/(i\tau_n) \to 0$. So $\mathcal{A}_0 z = 0$, contradicting Lemma 9.48. With Lemma 9.45, every quasimode therefore escapes: $1 \ll \langle \mathcal{L} z_n, z_n \rangle \lesssim |\tau_n|$ and $z_n \rightharpoonup 0$. - (b) **[COND]** on standard semiclassical-measure theory. In the mode variables of Theorem 9.47 the mass is a Cartesian pseudodifferential operator with symbol $\gamma|x|^2 \pm d\sqrt{\kappa_E}\,|\xi|^{-1}$ (plus the shear, of order $-1$). The characteristic curve $\Gamma = \{\gamma|x|^2|\xi| = d\sqrt{\kappa_E}\}$ is invariant under $x \to h^{1/2}x$, $\xi \to h^{-1}\xi$. The corresponding anisotropic measure of $(z_n)$ lives on $\Gamma$, since the equation is elliptic off it. - (c) At interior points of $\Gamma$, where the carrier $\xi_0$ is much larger than the blow-up frequency, the second-microlocal blow-up is Proposition 9.46(ii). Near $\xi_0$ the gauge $|D|^c$ is smooth and polynomially bounded, so the complex-Airy resolvent bound transfers and the measure vanishes there. The isotropic axis-core regime is Proposition 9.46(iii). That regime does not by itself cover concentration with $r|\xi| \to \infty$ while $r \to 0$; the global transport candidate below concerns precisely such a two-scale approach. $\Gamma$ never reaches $r = R$ at high frequency, by (iv). - (d) The region identified as (C*) is where the carrier is **comparable to the blow-up frequency**. There the gauge meets $\xi = 0$. The unrestricted complex-Airy Green operator does not preserve $\dot{H}^{-1/2}$ (Fourier weight $|\xi|^{-1}$): for suitable forcing supported away from zero its output has a nonzero trace at zero. **The inference that this prevents a quasimode lower bound is withdrawn.** Lemma 9.50 proves that bound on the full weighted graph domain; the failed Green operator tests range compatibility instead.

The proposed reduction targets **(C*)**: normalized window quasimodes carry no mass, uniformly, where carrier and blow-up frequency are comparable. Its normal-form estimate is now Lemma 9.50. Turning the model classification into an exhaustive argument still requires the quantitative localization and reconstruction statement** (Q)** below; an unspecified appeal to standard theory in (b) is not its proof.

**Lemma 9.50 (the normal form is bounded below through zero Fourier frequency) [EXACT, for the graph domain specified here].** Let $\nu > 0$, $\sigma \in \mathbb{R}\backslash\{0\}$ and $c = -\frac{1}{2} + i\omega$, $\omega \in \mathbb{R}$. Define

$$P_{\nu,\sigma,c} = -\nu\partial_x^2 + i\sigma(x + c\partial_x^{-1}).$$

Its whole-line realization is specified by the gauge, rather than by extending $1/k$ arbitrarily at zero. Put $\mathcal{H}_* = L^2(\mathbb{R}, |k|^{-1}dk)$, $v(k) = |k|^{-c}\widehat{a}(k)$, and

$$B_{\nu,\sigma} v = -\sigma v' + \nu k^2 v, \qquad D(B_{\nu,\sigma}) = \{v \in \mathcal{H}_* : v \in AC_{\mathrm{loc}}(\mathbb{R}\backslash\{0\}),\ B_{\nu,\sigma} v \in \mathcal{H}_*\}.$$

Then $a \in D(P)$ iff $v \in D(B)$, with $\widehat{Pa} = |k|^c Bv$. On this domain,

$$\boxed{\| P_{\nu,\sigma,c} a \|_2 \ \geq\ \nu^{1/3} |\sigma|^{2/3} \| a \|_2 .}$$

The bound is uniform in $\omega$, requires no Fourier support restriction away from zero, and excludes normalized quasimodes of the normalized whole-line normal form. It is a lower bound on the domain, not a claim of surjectivity for arbitrary $L^2$ forcing.

*Proof.* The gauge is an isometry from physical $L^2$ to $\mathcal{H}_*$:

$$\| v \|_{\mathcal{H}_*} = \| a \|_2 , \qquad \| Bv \|_{\mathcal{H}_*} = \| Pa \|_2 .$$

The factor $|k|^{-i\omega}$ is unitary. Reflection in $k$ handles the sign of $\sigma$, and $k = (|\sigma|/\nu)^{1/3} s$ preserves the measure $dk/|k|$. It therefore suffices to prove $\| Bv \|_{\mathcal{H}_*} \geq \| v \|_{\mathcal{H}_*}$ for $B = -\partial_s + s^2$.

First take $v \in C_c^\infty(\mathbb{R}\backslash\{0\})$. On $s > 0$, integration by parts gives the exact squared-norm identity

$$\| Bv \|_{*,+}^2 = \int_0^\infty \left( \frac{|v'|^2}{s} + s^3|v|^2 + |v|^2 \right) ds.$$

Since $v(0) = 0$, Cauchy–Schwarz and integration in $s$ give

$$\int_0^1 \frac{|v(s)|^2}{s}\,ds \le \frac{1}{4}\int_0^1 \frac{|v'(s)|^2}{s}\,ds, \qquad \int_1^\infty \frac{|v(s)|^2}{s}\,ds \le \int_1^\infty s^3|v(s)|^2\,ds.$$

Thus $\| Bv \|_{*,+} \ge \| v \|_{*,+}$. On $s < 0$ the first-order energy has the favorable sign:

$$\mathrm{Re}\langle Bv, v\rangle_{*,-} = \int_{-\infty}^0 \left(|s| + \frac{1}{2s^2}\right)|v|^2 ds \ge \frac{3}{2^{4/3}} \| v \|_{*,-}^2 ,$$

because $\min_{t>0}(t^2 + 1/(2t)) = 3/2^{4/3} > 1$. Cauchy–Schwarz and addition of the two half-line squared bounds prove the claim on the core.

The domain issue is essential. For any $v$ in the stated maximal domain, $v' = s^2 v - Bv$ belongs to $L^2(ds/|s|)$ near zero. It is integrable there, so both one-sided traces exist; $v \in \mathcal{H}_*$ forces both to be zero. Moreover, on either side,

$$|v(s)|^2 \le \frac{|s|^2}{2}\int_0^{|s|} \frac{|v'(\pm t)|^2}{t}\,dt.$$

Consequently a cutoff removing $|s| < \varepsilon$ has graph-norm commutator tending to zero: its derivative costs $O(\varepsilon^{-1})$, but the last bound makes that cost $o(1)$. Cutoffs at infinity cost $O(R^{-1}) \| v \|_{\mathcal{H}_*}$, and ordinary mollification on compact sets away from zero completes the graph-core approximation. The estimate therefore extends to the whole stated domain. These forced traces are at **Fourier frequency** $k = 0$, not additional physical data at $x = 0$. ▫

**Why Proposition 9.49(d)'s example does not give quasimodes.** In normalized variables, the solution selected by decay at $+\infty$ is

$$v(s) = e^{s^3/3}\int_s^\infty e^{-t^3/3} g(t)\,dt \qquad (s > 0).$$

Its membership in $\mathcal{H}_*$ requires the range compatibility

$$\int_0^\infty e^{-t^3/3} g(t)\,dt = 0.$$

A nonnegative nonzero $g \in C_c^\infty((1,2))$ violates it, so the unrestricted Green operator does not map $\mathcal{H}_*$ to itself. That fact survives. The inference that it prevents a lower bound is **[FAIL]**: Lemma 9.50 proves a bounded inverse on the range. The distinction is **arbitrary forcing versus residual of an admissible state**. No unbounded gauge has been used on an arbitrary tempered distribution.

**What this settles, and the remaining transfer (Q).** The zero-frequency obstruction is removed for the entire whole-line normal form, uniformly in Richardson number. Applying the lemma after a justified blow-up is legitimate. It does **not** justify the blow-up itself. To obtain (C*) for the original window quasimodes, the localization/reconstruction maps must retain a positive amount of physical norm and make the normalized model residuals small. One sufficient precise formulation, for the part not already excluded by other models, is*

$$a_{n,j} = T_{n,j} z_n \in D(P_{n,j}), \quad \liminf_n \sum_j \| a_{n,j} \|_2^2 > 0, \quad \sum_j \| q_{n,j}^{-1} P_{n,j} a_{n,j} \|_2^2 \to 0, \qquad q_{n,j} = \nu_{n,j}^{1/3}|\sigma_{n,j}|^{2/3}.$$

*Lemma 9.50 would make these proposed transfer conditions contradictory. They require control of retained norm, commutators and inherited endpoints; neither their scaling nor Proposition 9.49(b) constructs the maps. This independent-model formulation* ***(Q) is superseded for the compactness objective by Theorem 9.60****, which exhausts actual states using the outer reflection, the exact Dirac identity and the six-component radial system. Full frozen-window high-frequency compactness and (C*) are therefore proved. Surjectivity, remaining finite frequencies, the term $\kappa t$, height outflow and global patching are separate obligations.

*The global quasimode argument.* The complete calculations, domain distinctions and numerical implementation are recorded in Attachment C. The following results concern the frozen operator with homogeneous radial data; they do not restore $\kappa t$ or solve the height problem.

> **Proposition 9.51 (all quasimode mass concentrates at the axis) [EXACT for the full frozen-window operator].** Write $\mathcal{A}_0 = \gamma r^2 I + K$, with $\gamma \neq 0$, retaining both the pressure endpoint and the full shear. Then $K$ is Hilbert–Schmidt in $H$. Every hypothetical sequence from Proposition 9.49 satisfies
>
> $$z_n \rightharpoonup 0, \quad Kz_n \to 0 \text{ strongly}, \quad \int r^2 |z_n|_H^2 \to 0, \quad \langle \mathcal{L} z_n, z_n \rangle \to \infty,$$
> $$\frac{\langle \mathcal{L} z_n, z_n \rangle}{|\tau_n|} \to 0, \qquad \frac{\| \mathcal{L} z_n \|_H}{|\tau_n|} \to 0.$$
>
> Hence every fixed annulus $r \geq \delta > 0$ loses all its $H$ mass. This is not a rate of concentration and does not control boundary derivatives.

*Proof.* Testing against $D(\mathcal{L})$ and applying Lemma 9.48 makes every weak limit zero; compactness then gives $Kz_n \to 0$. The full imaginary-part identity is

$$\gamma \int r^2 |z_n|_H^2 = \tau_n^{-1} \mathrm{Im} \langle \mathcal{R}(\tau_n) z_n, z_n \rangle - \mathrm{Re} \langle K z_n, z_n \rangle \to 0.$$

Thus $r^2 z_n \to 0$ and $\mathcal{A}_0 z_n \to 0$ strongly, giving the last limit from the equation. The real part gives the energy divided by $|\tau_n|$ tending to zero. Any bounded-energy subsequence contradicts compact embedding and unit norm. The Hilbert–Schmidt kernels, including the endpoints, are displayed in Attachment C. ▫

**A global virial [EXACT].** With $Wz = (re_r + 3e, ru_r + u)$ and homogeneous Dirichlet traces,

$$\mathrm{Re} \langle \mathcal{A}_0 z, Wz \rangle_H = \gamma \left( \int_0^R r^3 |u|^2 dr + |\Psi(R)|^2 \right).$$

The pair terms cancel in the real part; the tilt and full shear give the displayed positive expression when $\gamma > 0$. The corresponding resolvent identity retains $\mathrm{Im} \langle \mathcal{L} z, Wz \rangle$ and differentiated boundary-layer contributions, so it is not yet a closed estimate. The calculation is given in Attachment C.

**Why a fine partition does not yet prove (Q) [BAL / method limitation].** At radius $\rho$ on the mode-zero curve, $|\xi_0| \asymp \rho^{-2}$. Comparability with $(|\tau|\rho)^{1/3}$ gives $|\tau| \asymp \rho^{-7}$ and local scale $q \asymp \rho^{-4}$. A cutoff of width $w$ has a pair-commutator contribution $|\tau| b / (\xi_0^2 w q) \asymp \rho / w$, whereas replacing $\gamma r^2$ by its tangent costs $|\tau|\,|\gamma| w^2 / q \asymp w^2 / \rho^3$. These estimates require both $w \gg \rho$ and $w \ll \rho^{3/2}$ to be small. They cannot close by uniform norm absorption. This is a failure of that localization estimate, not a counterexample to the resolvent; a global propagation argument must retain the motion along the curve.

> **Proposition 9.52 (the outer pressure trace reflects the inviscid modes) [EXACT for the mass equation].** Put $b = d\sqrt{\kappa_E}$, $\beta = b/|\gamma|$, and temporarily allow nonzero inviscid $e, u$ traces at $R$. The solutions of $\mathcal{A}_0 z = 0$ away from the axis have
>
> $$\Psi = \frac{r}{\beta} \left( A H_1^{(1)}(\beta/r) + B H_1^{(2)}(\beta/r) \right).$$
>
> Keeping $p(R) = 0$ in the undifferentiated axial row imposes $R\Psi'(R) - 2\Psi(R) = 0$, hence
>
> $$B = -\frac{H_0^{(1)}(\beta/R)}{H_0^{(2)}(\beta/R)} A, \qquad |B| = |A|.$$
>
> This inviscid calculation is extended to actual viscous primitive coefficients on a shrinking outer collar by Proposition 9.54 below. Proposition 9.56 then passes the reflection to the transport limit using the exact corrected equation.

**The global transport candidate [BAL].** In flat radial modal coordinates the mass-kernel amplitude is $r^{-1}$, with phases $\phi_{\pm}^{'} = \pm\beta/r^2$. Keeping diffusion and the shear gives the transport balances

$$\mathcal{T}_{\pm}a_{\pm} = \pm\tau\frac{\gamma^2}{b}(r^4 a_{\pm}^{'} + r^3 a_{\pm}) + \nu\beta^2 r^{-4}a_{\pm}, \qquad \nu = 2L.$$

For positive $\gamma, \tau$, set $\rho_{\tau} = (\nu b^3/(7\gamma^4\tau))^{1/7}$, $s = (\rho_{\tau}/r)^7$, $a_{\pm} = r^{-1}B_{\pm}(s)$. This turns the displayed transport operators exactly into $q_{\tau}s^{4/7}(\mp\partial_s + 1)$ on $\mathcal{K} = L^2(s^{-6/7}ds)$, where $q_{\tau} = \nu\beta^2\rho_{\tau}^{-4} \asymp \tau^{4/7}$. The derivation from the original operator is an asymptotic balance, not yet a uniform reduction of its domain and residuals.

> **Lemma 9.53 (the reflected transport model is uniformly invertible) [EXACT on the stated model domain].** For $|\zeta| = 1$, let
>
> $$\mathcal{T}B = \left(s^{4/7}(-B_{+}^{'} + B_{+}),\ s^{4/7}(B_{-}^{'} + B_{-})\right), \qquad B_{-}(0) = \zeta B_{+}(0).$$
>
> On locally absolutely continuous pairs in $\mathcal{K}^2$ with $\mathcal{T}B \in \mathcal{K}^2$, this boundary-value problem is bijective and $\| B \|_{\mathcal{K}^2} \le C \| \mathcal{T}B \|_{\mathcal{K}^2}$ with $C$ independent of $\zeta$.

*Proof.* For forcing $F$, the two components are

$$B_{+}(s) = e^s\int_s^{\infty} e^{-t}t^{-4/7}F_{+}(t)dt, \qquad B_{-}(s) = \zeta B_{+}(0)e^{-s} + \int_0^s e^{-(s-t)}t^{-4/7}F_{-}(t)dt.$$

The trace of $B_{+}$ is bounded by Cauchy–Schwarz, since $\int_0^{\infty} e^{-2t}t^{-2/7}dt < \infty$. On $(0,1)$ use integrability of $s^{-6/7}$ and the bound $|\int_0^s e^{-(s-t)}t^{-4/7}F(t)dt|^2 \le (7/5)s^{5/7}\| F \|_{\mathcal{K}}^2$. On $s, t \ge 1$, conjugation by the square root of the weight leaves an exponential convolution with additional factor $s^{-3/7}t^{-1/7} \le 1$. The remaining terms decay exponentially. These bounds give the asserted uniform inverse; the detailed domain check is in Attachment C. ▫

**Boundary countertest.** Without the reflection relation, $(0, e^{-s})$ is a nonzero kernel of this transport model. Therefore the global pressure relation cannot be discarded merely because fixed-annulus $H$ mass vanishes. It can survive the axis rescaling as a condition at infinity. This is not a kernel of the original Dirichlet resolvent.

**[NUM]** The full-window sparse calculation, independently matched to cv30's dense operator, gives $\sigma_{\min} = 3071.495,\ 41759.634,\ 574353.854$ at $\tau = 10^6, 10^8, 10^{10}$ with 8,000 nodes. The 4,000-node values differ by at most 0.061%. The ratios to $\tau^{4/7}$ are 1.14493, 1.12028, 1.10890; this supports the transport balance and proves no continuum bound.

> **Proposition 9.54 (outer reflection transfer for actual viscous states) [EXACT].** Keep the full frozen-window operator and its Dirichlet domain. Put $t = |\tau| \ge 1$, $f = \mathcal{R}(\tau)z$, $Y = (\Psi, p)^T$, and let $\mathsf{F}$ have columns $((r/\beta)H_1^{(j)}(\beta/r), (\gamma/d)H_0^{(j)}(\beta/r))^T$, $j = 1, 2$. The actual primitive coefficients $C = \mathsf{F}^{-1}Y$ satisfy $C_2(R) = \zeta C_1(R)$ exactly. For fixed $0 < a_0 \le \min(1, R/2)$ and $t^{-1/2} \le a \le a_0$,
>
> $$\sup_{a \le r \le R}|C(r) - C(R)| \le C_*\left(t^{-1/4}a^{-3} + t^{-3/4}a^{-5} + t^{-1}a^{-13/2}\right)(\| z \|_H + \| f \|_H).$$
>
> In particular, with $a_t = a_0 t^{-1/28}$ and $\rho_t = (\nu b^3/(7\gamma^4 t))^{1/7}$, for sufficiently large $t$,
>
> $$\sup_{a_t \le r \le R}\frac{|C_2(r) - \zeta C_1(r)|}{\sqrt{\rho_t}} \le C_* t^{-1/14}(\| z \|_H + \| f \|_H).$$

*Proof.* With $g = \mathcal{A}_0 z$, the exact primitive system is

$$Y^{'} = \mathsf{B}Y + \mathsf{D}g, \qquad \mathsf{B} = \begin{pmatrix} 2/r & -d/(\gamma r) \\ d\kappa_E/(\gamma r^3) & 0 \end{pmatrix}, \qquad \mathsf{D} = \begin{pmatrix} 0 & 1/(\gamma r) \\ \kappa_E/(2\gamma r) & 0 \end{pmatrix}.$$

Here $\mathsf{F}' = \mathsf{BF}$, $\det\mathsf{F} = -4i\gamma r^2/(\pi d\beta^2)$, and $C' = \mathsf{M}g$ for $\mathsf{M} = \mathsf{F}^{-1}\mathsf{D}$. The pressure trace gives the exact outer relation. Set $\mathsf{Q} = \mathsf{M}\mathrm{diag}(3,1)/r - \mathsf{M}'$. Substitution of $g = (f + \nu\mathrm{diag}(\Delta_4, \Delta_2)z)/(i\tau)$ and two integrations by parts give

$$C(R) - C(r) = \frac{1}{i\tau}\int_r^R \mathsf{M}f\, d\ell + \frac{\nu}{i\tau}[\mathsf{M}z' + \mathsf{Q}z]_r^R - \frac{\nu}{i\tau}\int_r^R \mathsf{Q}'z\, d\ell.$$

The graph and energy bounds are $\| \mathcal{L}z \| \le Ct(\| z \| + \| f \|)$ and $\| \mathcal{L}^{1/2}z \| \le Ct^{1/2}(\| z \| + \| f \|)$. Weighted collar estimates and one-dimensional interpolation bound the two bracket terms by $Ca^{-3}t^{3/4}(\| z \| + \| f \|)$ and $Ca^{-5}t^{1/4}(\| z \| + \| f \|)$. The coefficient bounds $|\partial_r^j \mathsf{M}_{\cdot e}| \le Cr^{-3/2-2j}$ and $|\partial_r^j \mathsf{M}_{\cdot u}| \le Cr^{-5/2-2j}$, $j \le 2$, bound the integrals by $Ca^{-5/2}\| f \|$ and $Ca^{-13/2}\| z \|$. These bounds, including the domain and Hankel details, are proved in Attachment C, §7. Substituting $a_t$ and dividing by $\sqrt{\rho_t}$ gives the result. ▫

**The remaining reconstruction defect [EXACT identity / OPEN estimate].** The same calculation gives

$$\tilde{C} = C - \frac{\nu}{i\tau}(\mathsf{M}z' + \mathsf{Q}z), \qquad \tilde{C}' = \frac{1}{i\tau}(\mathsf{M}f - \nu\mathsf{Q}'z).$$

No derivative of the residual occurs. Separately, with $\mathsf{W} = \mathsf{L}_0^{-1}\mathsf{F}$ and $\mathsf{L}_0 = \begin{pmatrix} \gamma r^4/(2d) & 0 \\ \gamma^2 r^4/d^2 & -\gamma r^2/d \end{pmatrix}$, the mass rows give the exact reconstruction

$$z = \mathsf{W}C + \frac{\mathcal{A}_0 z}{\gamma r^2}, \qquad r^2\mathsf{W}^*\mathrm{diag}(\kappa_E r^3/4, r)\mathsf{W} = \frac{4}{\pi\beta}I + O(r).$$

Thus $\sqrt{\rho_t}$ is the appropriate primitive-amplitude normalization at $r \asymp \rho_t$, but the weighted defect $r^{-2}\mathcal{A}_0 z$ must be controlled to retain physical norm in these coefficients. Proposition 9.51 only controls the unweighted mass residual. The corrected identity does not by itself make this defect, or $\tilde{C} - C$, small. See §8 of Attachment C.

> **Proposition 9.55 (actual local transport limits of corrected amplitudes) [EXACT].** Suppose $|\tau_n| \to \infty$, $z_n \in D(\mathcal{L})$ and $\| z_n \|_H + \| \mathcal{R}(\tau_n)z_n \|_H \le N$. Set $h_n = \rho_{|\tau_n|}$ and $c_n(x) = h_n^{-1/2}\tilde{C}_n(h_n x)$. For each compact interval $I \subset (0,\infty)$, $\| c_n \|_{H^1(I)} \le C_I N$ eventually. A subsequence converges locally uniformly and weakly in $H^1$ on compact intervals. With $\sigma = \mathrm{sgn}(\gamma\tau_n)$ constant on this subsequence, the limit satisfies
>
> $$c' = 7\sigma x^{-8}\mathrm{diag}(1,-1)c, \qquad \int_0^\infty x^{-2}|c|^2 dx \le \frac{\pi\beta}{4}\liminf_n \| z_n \|_H^2 .$$
>
> Thus $c = (Ae^{-x^{-7}}, 0)$ for $\sigma = 1$, with the components exchanged for $\sigma = -1$. This proposition alone does not retain the outer reflection; Proposition 9.56 supplies it and proves $A = 0$.

*Proof.* In the isometric coordinates $v = ((\sqrt{\kappa_E}/2)h^2 x^{3/2}e(hx), hx^{1/2}u(hx))$ and the corresponding forcing $w$, the corrected identity bounds $\| c' \|_{L^2(I)} \le C_I(\| v \|_2 + h^4 \| w \|_2)$. To bound its additive constant, use the exact reconstruction to express $c$ as bounded oscillatory matrices times $v, hv, h^2v', h^3v''$ and $h^5 w$. Each derivative of the matrix coefficients costs at most $h^{-1}$. Pairing with a fixed smooth interior test function and integrating by parts therefore bounds the mean of $c$ without assuming bounds on derivatives of $v$. The derivative and mean bounds give $H^1$ boundedness and local uniform subsequential convergence. The same integrations by parts reduce the corrected equation, in distributions on $I$, to

$$c_n' - \mathsf{R}_{h_n}c_n \to 0, \qquad \mathsf{R}_h = -\frac{\nu h}{i\tau}\mathsf{Q}'(hx)\mathsf{W}(hx) = 7\sigma x^{-8}\mathrm{diag}(1,-1) + O_I(h).$$

This identifies the limit equation. The leading coefficient extraction matrix has norm square $(\pi\beta/4)x^2 I + O_I(h)$, so weak lower semicontinuity and exhaustion give the weighted norm bound. It excludes the exponentially growing axis branch. The explicit matrices, integration bounds and normalization audit are in Attachment C, §9. ▫

> **Proposition 9.56 (the outer reflection passes to the limit) [EXACT].** Under Proposition 9.55's hypotheses, for $1 \le L \le R/h_n$,
>
> $$|c_{n,2}(L) - \zeta c_{n,1}(L)| \le CN\left(L^{-13/2} + h_n^4 L^{-5/2} + h_n^{5/4}\right).$$
>
> Every local transport limit is therefore zero. The full sequence $c_n$ converges locally uniformly to zero, $c_n(R/h_n) \to 0$, and $\int_\varepsilon^{R/h_n} x^{-2}|c_n|^2 dx \to 0$ for every fixed $\varepsilon > 0$.

*Proof.* The exact corrected equation gives, globally on $0 < x < R/h_n$,

$$|c_n'(x)| \le C\left(x^{-7}|v_n(x)| + h_n^4 x^{-3}|w_n(x)|\right).$$

Cauchy–Schwarz bounds its integrated tail from $L$ to the original outer endpoint by $CN(L^{-13/2} + h_n^4 L^{-5/2})$. At that endpoint the original primitive coefficients reflect exactly. Since $z_n(R) = 0$, the correction is $-\nu \mathsf{M}(R) z_n'(R)/(i\tau_n)$; the fixed-collar derivative trace bound gives normalized size $O(h_n^{5/4} N)$. This proves the reflection estimate. For the only possible limit in Proposition 9.55, it gives $|A|e^{-L^{-7}} \le CNL^{-13/2}$, so $A = 0$. The same derivative-tail estimate gives convergence of the outer coefficient values and the weighted norm on $x \ge \varepsilon$. Full details are in Attachment C, §10. ▫

> **Proposition 9.57 (physical mass escapes below the transport scale) [EXACT].** Under Proposition 9.55's hypotheses, for every fixed $\varepsilon > 0$,
>
> $$\| \mathbf{1}_{r \ge \varepsilon \rho_{|\tau_n|}} z_n \|_H \to 0.$$
>
> A normalized quasimode would therefore concentrate entirely at $r = o(|\tau_n|^{-1/7})$.

*Proof.* The exact differential reconstruction obtained from $\tilde{C}$ becomes, in the isometric coordinates of Proposition 9.55,

$$-\nu h^3 v'' - \nu(h^2 a x^{-2} J + 2h^3 x^{-1} P)v' + \nu(h\beta^2 x^{-4} I + h^3 x^{-2} D_0 + h^2 a x^{-3} K_0)v + i\omega x^2 v = i\omega x^2 \mathsf{B}_h c + h^5 w,$$

where $a = b/\gamma$, $\omega = \tau\gamma h^7$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P = \mathrm{diag}(0,1)$, $D_0 = \mathrm{diag}(3/4, -29/4)$, $K_0 = \begin{pmatrix} 0 & -7/2 \\ 9/2 & 0 \end{pmatrix}$ and $\mathsf{B}_h = h\,\mathrm{diag}(\kappa_E(hx)^3/4, hx)^{1/2}\mathsf{W}(hx)$. In particular $|\mathsf{B}_h c| \le Cx^{-1}|c|$ and $v(R/h) = 0$. Test with $\chi^2 x^{-2}\bar{v}$, where $\chi = 0$ below $\varepsilon/2$ and $\chi = 1$ above $\varepsilon$. The principal pairing is $\nu E_h + i\omega M_h$, with $E_h = h^3 \int \chi^2 x^{-2}|v'|^2$ and $M_h = \int \chi^2 |v|^2$. All other left-hand terms are bounded by $\delta E_h + C_{\varepsilon,\delta} h \| v \|_2^2$. Absorption and Young's inequality give

$$M_h \le C \int_{\varepsilon/2}^{R/h} x^{-2}|c|^2 dx + C_\varepsilon h \| v \|_2^2 + C_\varepsilon h^{10} \| w \|_2^2 .$$

Proposition 9.56 makes the first term vanish, and boundedness handles the other two. This tests a single exact differential identity; it does not cut off the Volterra operator or glue symmetrizers. The matrices and weighted estimate are derived in Attachment C, §11. ▫

> **Proposition 9.58 (the entire fixed axis-core scale is excluded) [EXACT].** For every fixed $B > 0$ and sufficiently large $t = |\tau|$,
>
> $$\| \mathbf{1}_{r \le Bt^{-1/6}} z \|_H \le C_B\left(t^{-2/3} \| \mathcal{R}(\tau) z \|_H + t^{-1/12} \| z \|_H\right).$$

*Proof.* In the isometric coordinates $y = ((\sqrt{\kappa_E}/2)re, u)$ with measure $rdr$, set $D = \begin{pmatrix} 0 & \partial_r \\ -\partial_r - r^{-1} & 0 \end{pmatrix}$ and $D_{\mathrm{m}} = D - 2r^{-1}E_{21}$. Differentiation of the full original mass gives the exact identity

$$\left(\nu D^3 + i\tau b\right) y + i\tau\gamma r^2 D_{\mathrm{m}} y = D\widehat{f}.$$

On the whole regular radial half-line $D$ is self-adjoint, and the explicit Fourier inverse satisfies $\| (\nu D^3 + i\tau b)^{-1} D^j \| \le C_j t^{j/3-1}$ for $0 \le j \le 3$. A cutoff at $\alpha t^{-1/6}$, with fixed small $\alpha$, absorbs the tilt and gives an inner error $O(t^{-1/6}) \| y \|$. In the rest of the fixed core, flat-coordinate localization on intervals of width $t^{-1/4}$ uses the constant inverse of $\nu(J\partial_r)^3 + i\tau(bI + \gamma\rho^2 J\partial_r)$. Its scaled Fourier multipliers have squared modulus $\nu^2 s^6 + (b + qs)^2$, uniformly bounded away from zero for bounded $q = \gamma\rho^2 t^{1/3}$. All cutoff and freezing errors are $O_B(t^{-1/12})$. Derivatives are reordered onto the inverse, so only the original state and forcing norms occur. The explicit commutators, domains and bounded-overlap summation are in Attachment D, §§2–3. ▫

> **Lemma 9.59 (uniform estimate on the intervening interval) [EXACT].** Put $U = ((\sqrt{\kappa_E}/2)r^{3/2}e, r^{1/2}u)$, $Z = (b\Psi/r^{5/2}, dp/r^{3/2})$, $V = U'/(\sqrt{t}\, r)$ and $X = (Z, U, V)$. There are $K_0, r_0, \eta_0, C > 0$ such that, on every interval $[a, c] \subset (0, r_0]$ with $\sqrt{t}\, r^3 \ge K_0$ and $tr^7 \le \eta_0$,
>
> $$\int_a^c |X|^2 dr \le C\left( a|X(a)|^2 + c|X(c)|^2 + \int_a^c r^7 |F|^2 dr \right),$$
>
> where $F$ is the forcing in the flat physical coordinates.

*Proof.* The original equations give an exact six-component first-order system with principal matrix $r^{-2}A(K, r)$, $K = \sqrt{t}\, r^3$, and bounded matrix terms of size $r^{-1}$. Its six eigenvalues are the roots of

$$\nu\lambda^3 - i\mathrm{sgn}(\tau)\gamma K^2\lambda - i\mathrm{sgn}(\tau)K^2\mu_{\pm} = 0, \qquad \mu_{\pm} = -\gamma r \pm i\sqrt{b^2 - \gamma^2 r^2}.$$

For $K$ large and $rK^2 = tr^7$ small, the two slow real parts have opposite signs and absolute value at least $cK^{-2}$; the four fast real parts have absolute value at least $cK$, two of each sign. Explicit eigenvectors give a uniformly bounded diagonalizer and inverse, whose radial derivative is $O(r^{-1})$. In those coordinates use the constant signature of the six real parts and the radial weight $r$. The resulting differential inequality has positive term $c(K^2 r)^{-1}|X|^2$; it absorbs the coefficient derivatives because $K^2 r \le \eta_0$. The forcing remainder is $Cr^7|F|^2$. Integration gives the estimate. The exact system, root expansions, uniform conditioning and endpoint form are proved in Attachment D, §§4–5. ▫

> **Theorem 9.60 (full high-frequency compactness of the frozen window) [EXACT].** If $|\tau_n| \to \infty$, $z_n \in D(\mathcal{L})$ and $\| z_n \|_H + \| \mathcal{R}(\tau_n) z_n \|_H$ is bounded, then $\| z_n \|_H \to 0$. In particular,
>
> $$\inf_{\substack{z \in D(\mathcal{L}) \\ \| z \|_H = 1}} \| \mathcal{R}(\tau) z \|_H \to \infty \quad \text{as } |\tau| \to \infty.$$
>
> This excludes normalized quasimodes on the original domain, including the former (C*) regime. It does not assert the numerical 4/7 rate or surjectivity.

*Proof.* Choose fixed large $B$ with $B^3 \ge K_0$, and fixed small $\delta$ with $t(2\delta\rho_t)^7 \le \eta_0$. Proposition 9.58 gives vanishing physical norm on every fixed core ball $r \le Bt^{-1/6}$. To control the full endpoint state $X$ there, observe the six-dimensional system on intervals of length $t^{-1/3}$. Its limiting coefficient matrices form a compact family. Observation of $U$ is injective: $U \equiv 0$ forces $V = 0$, then the $V$ equation forces $Z = 0$. The finite-dimensional observation Gramian is therefore uniformly positive on that family. Duhamel's formula gives $\int |X|^2 \le C_B(\int |U|^2 + t^{-4/3}\int |F|^2)$ on a core annulus and its enlargement. Thus an endpoint $a_n \asymp Bt_n^{-1/6}$ can be chosen with $a_n|X_n(a_n)|^2 \to 0$.

On $r \asymp \delta\rho_t$, Proposition 9.57's energy estimate controls both $U$ and $V$. The exact correction between $C$ and $\tilde{C}$, together with Proposition 9.56, also controls $Z$. Averaging gives an endpoint $c_n \asymp \delta\rho_{t_n}$ with $c_n|X_n(c_n)|^2 \to 0$. Lemma 9.59 removes the norm on $[a_n, c_n]$; its forcing contribution tends to zero since $c_n^7 \| F_n \|_2^2 \to 0$. Proposition 9.58 removes the region below $a_n$, and Proposition 9.57 removes the region above $c_n$. These intervals exhaust the physical norm. All endpoint observation and reconstruction estimates are given in Attachment D, §6. ▫

**Global status [EXACT for the frozen-window high-frequency statement].** Theorem 9.60 closes full (C*) and the stated large-frequency target (S′a). The earlier independent-normal-form transfer formulation (Q) is superseded by this direct exhaustion of actual states. The pressure endpoint, core, fast modes, intermediate interval and physical norm are all retained. Surjectivity, possible resonances on the remaining compact frequency interval, the source-fold term $\kappa t$, height outflow, variable-profile transfer and global inboard construction are separate obligations.

*Obstruction tests on the compactness step. - (P) Davies–Dencker–Sjöstrand–Zworski pseudomodes.* Treating the shear's imaginary mass shift $iH_{c,X}/(2\xi_0)$ as principal gives real characteristic points with a sign-changing Poisson bracket, which would mean exponentially good quasimodes. **Fails:** it is the same inconsistency as in Proposition 9.39(b). The shift is locked to the tilt, the whole-line gauge removes it exactly, and a gauge of polynomial size cannot create exponential pseudomodes. Consistently, $\sigma_{\min}$ grows like $\tau^{0.56}$ (cv37). - *(K0) Zero-frequency gauge failure:* **excluded as an obstruction to the whole-line lower bound** by Lemma 9.50. Range compatibility survives; Theorem 9.60 supplies the required global compactness through a direct operator argument. - *(B) Boundary:* fails, as in Proposition 9.46(iv).

**Status of the axis fold.** - (T1) is closed (Theorem 9.40). - (T2) and (T3) are closed for the symmetric fold (Theorem 9.41). - (S) is [FAIL / REDUNDANT]. - The stated large-frequency target (S′a) is **[EXACT]** by Theorem 9.60, as is full (C*). The lower bound on the original frozen-window domain tends to infinity. Theorem 9.60 supplies actual norm exhaustion, including the pressure endpoint, core and intervening interval. It does not assert surjectivity, an all-frequency bound, or the numerical 4/7 rate.

Numerically $\sigma_{\min} \sim \tau^{0.56}$. **No obstruction has been found.** Condition (G) is unaffected.

> **Compactness step completed.** Theorem 9.60 excludes every normalized high-frequency quasimode of the full frozen-window operator. The next obligations concern the remaining finite frequencies and surjectivity, then the source-fold term, height outflow and transfer to the unfrozen global construction. No $|\tau|^{4/7}$ lower-bound rate is claimed by this qualitative proof.

### 9.16.9 The global midplane preparation: lift, bridge, slaving obstruction, and the missing step (made redundant by (h))

Theorem 9.18 orders the work: the midplane must be completed inboard before anything in height. This subsection addresses that completion with $v := U(\cdot, 0)$ restored, and records what it proves, what it rules out, and the single step it leaves.

**(a) The midplane state with $v \neq 0$ [EXACT | TRANSCRIPTION].** From (4.8)–(4.9) at $\eta = 0$, with $E_1 := \partial_\eta E|_0$, $U^{(1)} := \partial_\eta U|_0$ and $\Pi_1 := \partial_\eta \Pi|_0$,

$$S_q|_0 = -Wl - h - v\frac{E_1}{E_0}, \qquad S_n|_0 = -W D_X v - Av - vU^{(1)} - \Pi_1, \qquad W = 1 - A_X\left(U^{(1)}\right), \qquad \Pi_1 = -\int_X^\infty \frac{E_0 E_1}{x}\, dx.$$

> **Proposition 9.19.** The midplane stress depends exactly on $(E_0, v, U^{(1)}, E_1)$. $\Pi_0$, $A_X(v)$, $U^{(2)}$ and $E_2$ do not enter; $\Pi_1$ is determined by $E_0E_1$. $E_1$ enters the angular source only multiplied by $v$, and the axial source only through $\Pi_1$. *(Verified symbolically.)*

Under $v \equiv 0$ the jet $E_1$ is invisible to the angular source, which is why §9.16.8(a) could force it to zero. With $v \neq 0$ it is a genuinely independent state.

**(b) Lemma 9.12 is a statement about a class [EXACT | TRANSCRIPTION].**

> **Lemma 9.20.** At a point where $l = 0$ and the demand vanishes identically nearby, $l' = \frac{1}{2}\left(h + vE_1/E_0\right)$.

So Lemma 9.12 — no stress-free downward crossing — holds exactly where $vE_1 = 0$. In the class $E_1 = 0$ the passage through $0 \le l < 1$ must carry nonzero demand whatever $v$ is. Where $vE_1/E_0 < -h$, a stress-free crossing is not excluded. The statement that Lemma 9.12 is independent of $v$ is true only in the first class.

**(c) The source lift, without division [EXACT | TRANSCRIPTION, class $E_1 = 0$].** Write $\Psi := \int_0^X U^{(1)} dx = X\,A_X(U^{(1)})$. The angular source is affine in $\Psi$ with coefficient $l$, which degenerates exactly at the crossing. Parametrising by a free smooth function $m$ removes the degeneracy.

> **Theorem 9.21 (source lift).** Let $l$ be smooth with $l(0^+) = 1$, let $m$ be smooth, and on any interval where $a = 2 - 2l > 0$ define $q$ by
>
> $$q' = \frac{2(1+q)\,l' - h + l\,m}{a}, \qquad \Psi := X(m+1) + a(1+q).$$
>
> Then the demand has $w_1 = q\,a$ exactly, with $U^{(1)} = \Psi' = (1+l)m + (1-h) + Xm'$. The axial demand is
>
> $$E\,w_2 = 2Xv' - XWv + D\int_0^X v\,dx - 2\int_0^X U^{(1)} v\,dx,$$
>
> with no division and no loss of derivatives beyond $v'$. The stress-free collar is $q = 0$, $m = (h - 2l')/l$; the crossing compatibility $q'(X_0) = (1+q)l' - h/2$ holds automatically; and wherever $q = 0$, $m = -1$ and $l = -h$, $\Psi = 2 + \alpha$ — **the join to the layer delivers the flux constant of Corollary 9.8 by itself.**

*Proof.* With $w_1 = qa$, $D_X w_1 + l w_1 = \Sigma_q$ and $\Sigma_q = -X(l+h) + l\Psi - D_X a - la$ give $l\Psi = X\left[q'a - 2(1+q)l' + l + h\right] + la(1+q)$. The equation for $q$ turns the bracket into $X(lm + l)$, so $l\Psi = l\left[X(m+1) + a(1+q)\right]$. The axial formula is one integration by parts of $\int_0^X S_n$, the boundary terms vanishing by axis regularity. ▫ *(Both identities verified symbolically; the lift reproduces the source-computed $w_1$ to $7 \times 10^{-10}$.)*

**(d) The slaving obstruction [EXACT | TRANSCRIPTION, class $E_1 = 0$].**

> **Theorem 9.22.** Suppose $w_2 = -\tilde{q}\,b_s$ on $(0, X_*]$ for a smooth $\tilde{q} > -1$. Then $v$ solves
>
> $$2\left((1+\tilde{q})Xv'\right)' - XWv' - \left(A + U^{(1)}\right)v = 0 \qquad \text{on } (0, X_*],$$
>
> and if $v(X_*) = v'(X_*) = 0$ then $v \equiv 0$ on $(0, X_*]$.

*Proof.* $E(w_2 + \tilde{q}\,b_s) = 2(1+\tilde{q})Xv' - XWv + D\int_0^X v - 2\int_0^X U^{(1)} v$ vanishes identically, and its derivative is the displayed operator (verified symbolically). The equation is linear and regular on $(0, X_*]$, with leading coefficient $2(1+\tilde{q})X > 0$, so zero Cauchy data at $X_*$ give the zero solution. ▫

Joining the layer requires $v(X_a) = v'(X_a) = 0$. Hence, in the class $E_1 = 0$, **an exactly on-axis global preparation** $w = q\,s$ **is impossible** (take $\tilde{q} = q$), and so is a preparation whose axial stress vanishes throughout (take $\tilde{q} = 0$). Every such preparation contains a *return window* on which the axial stress is not slaved to the axial shear. The on-axis ansatz earns its keep locally and fails globally, for an exact reason.

> **Lemma 9.23 (onset).** If the demand switches on at $X_c$ with $0 < l(X_c) < 1$, then $|b_s(X_c)| \ge 2\sqrt{l(1-l)} > 0$. So $v \not\equiv 0$ on the stress-free collar, where, in the class $E_1 = 0$, it is a multiple of the axis-regular solution of the $\tilde{q} = 0$ equation.

**(e) The shear bridge [EXACT | TRANSCRIPTION].**

> **Theorem 9.24.** In Theorem 9.21 let $m \geq 0$ and $m' \geq 0$ on $(0, X_0]$, where $l(X_0) = 0$, and let $q > 0$ on $(X_c, X_0]$. Let $v_{\mathrm{AX}}$ be the axis-regular solution of Theorem 9.22's equation with $\tilde{q} = q$ and $v_{\mathrm{AX}}(0) = 1$. Then $v_{\mathrm{AX}} > 0$ and $v'_{\mathrm{AX}} > 0$ on $(0, X_0]$, and for $c$ large $v = c\, v_{\mathrm{AX}}$ satisfies (CROSS), $|b_s| > 2\sqrt{l(1-l)}$, on $[X_c, X_0]$ with any prescribed margin. The demand $w = q\, s$ is then strictly cone-admissible at every radius of $(X_c, X_1)$, for any $X_1$ up to which $v = c\, v_{\mathrm{AX}}$ is kept: $M_1 = q|s|^2$, $M_2 = 2aq^2|s|^4$, and $M_0 > 0$ by (CROSS) where $l \geq 0$ and automatically where $l < 0$.

*Proof.* $U^{(1)} = (1+l)m + (1-h) + Xm' \geq 1 - h > 0 > -A$. At the axis $v'_{\mathrm{AX}}(0) = \left(A + U^{(1)}(0)\right)/2 > 0$. At a first zero of $v'$ with $v > 0$ the equation gives $2(1+q)Xv'' = \left(A + U^{(1)}\right)v > 0$, so $v'$ cannot reach zero from above. $E > 0$ and $v'_{\mathrm{AX}}$ bounded below on the compact $[X_c, X_0]$ give $c$. ▫

**[NUM]** $\alpha = 0.2$; $l$ smooth and monotone from 1 to $-h$ on $(0.5, 10)$; onset at $l = 0.8$; $m$ as in Theorem 9.24. (CROSS) margin 0.37, $\min(U^{(1)} + A) = 3.7$. The source-recomputed demand lies on the cone axis, $M_1/(|w||s|) = 1$, with angular margin 0.45 rad after onset, 0.59 before the crossing and $\geq 0.027$ rad through the crossing onto the plateau — points below the quadrature noise floor, where $q \leq 0.03$, being covered by the theorem rather than by the numerics. The join values $l = -h$, $v = b_s = 0$, $w = 0$, $\Psi = 2 + \alpha$ are reproduced exactly.

**(f) The return window: what was tried, and why it is hard [FAIL + NUM].** Three families were tested, each of the form $v = c\, v_{\mathrm{AX}}\, \varphi$ with $\varphi$ a one- or two-stage smooth cutoff after the crossing, and with the return condition $Ew_2 = 0$ enforced exactly (it is $\int \varphi' G\, dx = 0$, $G := 2(1+q)Xv'_{\mathrm{AX}} - XWv_{\mathrm{AX}}$): $q \approx 0.6$; $q \approx 8$ from a stronger pre-crossing drive; and the zero of $G$ steered by a local bump in $m$. **All three rotate the demand out of the cone inside the window.** The mechanism is visible in the output. During a steep return the integrated part of $w_2$ lags the shear, $w_2 \approx b_s$, so $M_1 \approx qa^2 - b_s^2$, and admissibility needs roughly $|b_s| \lesssim \min\{a\sqrt{q},\ 3.2\, qa/(1+q)\}$. A gentle return needs $G \approx 0$ across a long window, and steering $G$ costs $q$. This is evidence about these families. It is **not** an obstruction, and none is claimed.

**(g) The first-order odd part of $E$ supplies the missing control [EXACT | TRANSCRIPTION].** Theorem 9.22 bites because, with $E_1 = 0$, the axial demand has no control independent of $v$ except $U^{(1)}$, which the angular side has already spent. Let $E_1 \neq 0$ be supported in a window where $l < 0$, and put $e_1 := E_1/E_0$ and $g := Xe_1/l$. The angular source changes only by $-Xve_1$, which is absorbed exactly by $\Psi \mapsto \Psi + vg$. The on-axis condition $w_2 = -q\, b_s$ then becomes

$$v^2 g' - \int_X^\infty \frac{E_0^2\, l\, g}{y^2}\, dy = \mathcal{L}_q^0[v], \qquad \text{equivalently} \qquad \left(v^2 g'\right)' + \frac{E_0^2\, l}{X^2}\, g = \left(\mathcal{L}_q^0[v]\right)',$$

where $\mathcal{L}_q^0$ is Theorem 9.22's operator with the $E_1 = 0$ coefficients. The right side is explicit once a return profile $v$ is chosen, and the equation is **inhomogeneous**, so Theorem 9.22 does not apply to it. *If* it has a solution $g$ vanishing with $g'$ at both ends of the window and satisfying $\int E_0^2\, l\, g\, y^{-2}\, dy = 0$, the window is exactly on-axis and $M_1 = q|s|^2$, $M_2 = 2aq^2|s|^4$ hold there automatically. These are finitely many linear conditions on a free return profile, and the equation degenerates only where $v$ itself returns to zero.

> **The Return Lemma [OPEN; REDUNDANT after (h)].** There exist a window $(X_1, X_r) \subset \{l < 0\}$, a smooth return profile $v$ with $v = c\, v_{\mathrm{AX}}$ near $X_1$ and $v \equiv 0$ near $X_r$, and a smooth $E_1$ supported in the window, such that the demand is strictly cone-admissible on the window and $Ew_2(X_r) = 0$.

With it, Theorems 9.21 and 9.24 and a flat decay of $q$ with $v = 0$ — where $w = (qa, 0)$ is on-axis and $\Psi \to 2 + \alpha$ — give the **Global Midplane Preparation**: $w = 0$ on the axis collar, $w \in \mathrm{int}\mathcal{C}$ through the preparation, $w = 0$ at $X_a$, and Theorem 9.13's layer beyond. Two scope remarks. First, by Lemma 9.20 a stress-free crossing is not excluded when $vE_1/E_0 < -h$; whether an entirely stress-free preparation exists

— which would place the whole inboard region in the core in the sense of (A1b) — is settled in (h). Second, $E_1 \neq 0$ on the midplane alters the first-order jets of §9.16.8, which assumed $E_1 = 0$; the corner analysis concerns the layer, where $E_1 = 0$ is kept.

**(h) The stress-free route closes the midplane [EXACT | TRANSCRIPTION].** Run adversarially, the competing route does not fail; it removes the cone from the inboard problem altogether. With $e_1 := E_1/E_0$, the demand vanishes identically, $w_1 \equiv w_2 \equiv 0$, exactly when

$$l\,\Psi = N_0 + Xv\,e_1, \qquad N_0 := X(l+h) - 2Xl' + 2l(1-l),$$

$$\Pi_1 = P := 2(Xv')' - (X-\Psi)v' - Av - \Psi' v, \qquad e_1 = \frac{X\,P'}{E_0^2}, \qquad P(\infty) = 0.$$

The first relation is $\Sigma_q = 0$, the second is $\Sigma_n = 0$ written as a formula for $\Pi_1$, and the third is $\Pi_X = E^2/2X$ at first order in $\eta$, normalized at radial infinity. Eliminating $e_1$:

> **Theorem 9.25 (stress-free lift).** Let $l$ be smooth with $l(0^+) = 1$, crossing zero, and equal to $-h$ on $[X_s, \infty)$. Let $v$ be smooth with $v(0) \neq 0$, nonvanishing on $[0, X_s)$, and $v \equiv 0$ on $[X_s, \infty)$. A smooth axis-regular midplane profile with these $(l, v)$ and **zero demanded stress at every radius** exists if and only if the linear equation
>
> $$\frac{X^2v^2}{E_0^2}\Psi'' + \Big(l - \frac{X^2 v\,v''}{E_0^2}\Big)\Psi = N_0 + \frac{X^2 v}{E_0^2}\Big(2Xv''' + (4-X)v'' - (1+A)v'\Big)$$
>
> has a solution with $\Psi(0) = 0$ that is smooth at $X_s$. In that case $\Psi = N_0/l = 2 + \alpha$ on $[X_s, \infty)$ automatically; $E_1 = XP'/E_0$ is smooth and supported in $[0, X_s]$; $c_Q = c_N = 0$ at $\eta = 0$; and at the crossing $v\,e_1 = 2l' - h$, as Lemma 9.20 requires.

*Proof.* The three relations are equivalent to $\Sigma_q \equiv 0$, to $D_X(Ew_2) \equiv 0$ with $Ew_2(0) = 0$, and to the first-order pressure identity. Substituting $P' = R_v' + \Psi v'' - \Psi'' v$, with $R_v := 2(Xv')' - Xv' - Av$, into the first relation gives the equation. Axis regularity fixes both integration constants at zero, so the demand itself vanishes, not merely its derivative. Where $v \equiv 0$ the equation is algebraic, $l\Psi = N_0$, and at $l = -h$ this is $\Psi = 2 + \alpha$. ▫

The equation has **no singularity at the crossing** — $l$ enters only as a zeroth-order coefficient. Its homogeneous part becomes self-adjoint after one substitution. With $\Phi := \Psi/v$ on $(0, X_s)$,

$$\frac{X^2v^2}{E_0^2}\Psi'' + \Big(l - \frac{X^2 v v''}{E_0^2}\Big)\Psi = \frac{X^2 v}{E_0^2}\Big[\big(v^2\Phi'\big)' + q\,\Phi\Big], \qquad q := \frac{E_0^2\,l}{X^2},$$

a Sturm–Liouville operator with weight $v^2$ and **indefinite** potential $q$: positive before the crossing, negative after it.

*The axis.* $E_0^2 = H^2/2X$ with $H \sim cX$ gives $q \sim c^2/(2X)$, a regular singular point with Frobenius exponents 0 and 1 for $\Phi$. The exponent-0 branch carries a logarithm and has $\Psi(0) = v(0) \neq 0$. The condition $\Psi(0) = 0$ selects the exponent-1 branch, which is smooth: **one datum**. A smooth particular solution with $\Psi(0) = 0$ exists because the forcing vanishes at the axis.

*The return point.* Near $X_s$, $\kappa := X^2v^2/E_0^2 \to 0$ with all its derivatives, and $m := l - X^2 v v''/E_0^2 \to -h < 0$. The limiting equation is algebraic, $m\Psi = f$, which beyond $X_s$ reads $\Psi = N_0/l = 2 + \alpha$. The homogeneous modes are of Liouville–Green type, $\exp\big(\pm \int \sqrt{|m|/\kappa}\big)$: one is flat at $X_s$, the other blows up faster than any power. A flat cutoff makes $X_s$ an irregular singular point of infinite order, but it does not change this dichotomy, provided the cutoff is **tame**: $|v^{(j)}| \le C_j\, v\,(X_s - X)^{-2j}$, which holds for the $e^{-1/\xi}$ step used here. Smoothness through $X_s$ is exactly the absence of the blowing-up mode — **one scalar condition** — and a homogeneous solution that is smooth through $X_s$ is flat there.

**Theorem 9.26 (non-resonance and existence).** Under the hypotheses of Theorem 9.25, let $l > 0$ on $(0, X_0)$ and $l < 0$ on $(X_0, X_s]$, and let the cutoff of $v$ be tame. If

$$\min_{[0,X_0]} v^2 \;>\; Q := \int_0^{X_0} \frac{E_0(x)^2\, l(x)}{x}\, dx,$$

then no nonzero homogeneous solution is both axis-regular and smooth through $X_s$, and the equation of Theorem 9.25 has exactly one solution with $\Psi(0) = 0$ that is smooth through $X_s$. Hence a smooth, axis-regular, stress-free midplane preparation exists, and **the global midplane closes**.

*Proof.* **Non-resonance.** Let $\Psi$ be such a homogeneous solution and put $\Phi = \Psi/v$. Multiplying $(v^2\Phi')' + q\Phi = 0$ by $\Phi$ and integrating over $(\varepsilon, X_s - \delta)$ gives $[v^2\Phi\Phi'] = \int v^2\Phi'^2 - \int q\Phi^2$. The boundary term equals $\Psi\Psi' - \Psi^2 v'/v$. It vanishes at the axis, where $\Psi = O(X)$, and at $X_s$, where $\Psi$ is flat and $v'/v$ grows at most polynomially. Hence

$$\int_0^{X_s} v^2\Phi'^2 + \int_{X_0}^{X_s} |q|\,\Phi^2 = \int_0^{X_0} q\,\Phi^2.$$

Since $\Phi(0) = 0$, $\Phi(x)^2 \le x\int_0^{X_0}\Phi'^2$ on $[0, X_0]$, so the right side is at most $Q\int_0^{X_0}\Phi'^2 \le \left(Q/\min_{[0,X_0]} v^2\right)\int_0^{X_0} v^2\Phi'^2$. Under (NR) this forces $\Phi' \equiv 0$ on $(0, X_0)$. So $\Phi \equiv 0$ there, and uniqueness for the equation, which is regular on $(0, X_s)$, gives $\Psi \equiv 0$.

**Existence.** Let $g := f/m$ near $X_s$ — smooth, and equal to $2 + \alpha$ beyond $X_s$ — let $\chi$ be a cutoff equal to 1 near $X_s$, and write $\Psi = g\chi + v\phi$. Then $(v^2\phi')' + q\phi = \tilde{S}$, with $\tilde{S}$ bounded and flat at $X_s$ (there it equals $-vg''$). Let $V := \{\phi:\ \phi(0^+) = 0,\ \int v^2\phi'^2 + \int_{X_0}^{X_s}|q|\phi^2 < \infty\}$. On $V$ the same Hardy bound gives

$$\int v^2\phi'^2 - \int q\,\phi^2 \;\ge\; \left(1 - \frac{Q}{\min\limits_{[0,X_0]} v^2}\right)\int v^2\phi'^2 + \int_{X_0}^{X_s}|q|\,\phi^2,$$

so the form is coercive under (NR). The load $\zeta \mapsto \int \tilde{S}\zeta$ is bounded on $V$, by Poincaré where $v$ is bounded below and by $|q| \ge c$ near $X_s$. Lax–Milgram gives a unique weak solution, classical on $(0, X_s)$. At the axis, $\phi(0) = 0$ excludes the logarithmic branch; at $X_s$, finite energy excludes the blowing-up mode. Flatness of $\psi := v\phi$ follows by a barrier argument. With $a_1 = -\kappa g''/m$ and $a_{k+1} = -\kappa a_k''/m$, the remainder $r = \psi - \sum_{k \le n} a_k$ solves $r'' - \omega^2 r = -a_n''$ with $\omega^2 = -m/\kappa$. Tameness makes $B = 2\kappa G/c$ and $u = \kappa(X_s - X)^P$ supersolutions, for any flat majorant $G$ of the right side and any $P$, so the maximum principle gives $r = O\left(\kappa\,(X_s - X)^P\right)$. Differentiating the equation repeats the argument for every derivative. So $\Psi$ extends smoothly across $X_s$ by $2 + \alpha$. ▫

**Resonance is real below the threshold [EXACT | TRANSCRIPTION + NUM].** Write $v = \lambda\widehat{v}$. The homogeneous problem is then $(\widehat{v}^2\Phi')' + \lambda^{-2} q\,\Phi = 0$, so **resonance occurs exactly when $\lambda^{-2}$ is a positive eigenvalue of an indefinite singular Sturm–Liouville problem.** The exact resonance functional is the Wronskian

$$\mathcal{W}[l, v] := \Psi_a\Psi_s' - \Psi_a'\Psi_s,$$

which is constant in $X$; here $\Psi_a$ is the axis-regular branch and $\Psi_s$ the branch flat at $X_s$. Theorem 9.25 is uniquely solvable if and only if $\mathcal{W} \ne 0$. At a zero, $\partial_{\lambda^{-2}}\mathcal{W} = c\int q\,\Phi_a^2$ with $\int q\,\Phi_a^2 = \lambda^2\int \widehat{v}^2\Phi_a'^2 > 0$, so every resonance is simple and **transversal** in $\lambda$, and (NR) says there is none with $\lambda^2 > Q/\min\widehat{v}^2$. **[NUM]** For the design below, $Q = 2.2988$ and $\lambda_* = 1.5162$. A scan of $\mathcal{W}$ finds simple resonances at $\lambda = 1.2348,\ 0.5492,\ 0.3573,\ 0.2651,\ 0.2107, \ldots$, accumulating at 0; the largest lies 19% below the threshold. At $\lambda = 1.2348$ the finite-difference solve degenerates: $\max|\Psi|$ jumps from $7 \times 10^2$ at $\lambda = 1.2$ to

$5 \times 10^5$, the axis condition fails, and the source-recomputed stress rises from $9 \times 10^{-4}$ to 0.39. The obstruction test therefore succeeds below the threshold and is excluded above it, and an arbitrarily small change of amplitude removes any resonance.

**[NUM]** $\alpha = 0.2$; $l$ as in (e), crossing at $X_0 = 7.416$; $v = v_0$ on $[0, 11]$ with a $C^\infty$ return to 0 on $[11, 14]$. The equation was solved by finite differences on $4 \times 10^5$ points for $v_0 = 0.3,\ 1,\ 3$, and on $8 \times 10^5$ points for $v_0 = 1$. In every case $\Psi(X_s) = 2.2000000$; $\Psi'$ and $\Psi''$ have no jump across $X_s$; $v\,e_1$ at the crossing equals $2l' - h$ to $3 \times 10^{-5}$; and $e_1 \equiv 0$ beyond $X_s$. The demand recomputed **from the source definitions** — with $\Pi_1$ from its own quadrature of $E_0 E_1 / x$, not from $P$ — satisfies

$$\max|w_1| \le 5 \times 10^{-4}, \qquad \max|w_2| \le 1.4 \times 10^{-6},$$

against $\max|w| = O(10^1\text{–}10^2)$ for the same profile with $E_1$ set to zero. The three amplitudes are consistent with Theorem 9.26: $v_0 = 3$ lies above its threshold, while $v_0 = 0.3$ and $v_0 = 1$ lie below it at non-resonant amplitudes.

**What it buys, and what it costs.** It buys the global midplane: stress-free from the axis to $X_s$, then Theorem 9.13's layer at any $X_a \ge X_s$, then the exterior, so the demand is zero or strictly cone-admissible at every radius. The base preparation of Theorem 9.14 is no longer needed, since $w_1(X_a) = 0$ now holds with no undershoot, and (A1b) is realized literally: the whole inboard region carries no stress. The Return Lemma is **[REDUNDANT]**. The cost is size. The solutions have $|\Psi|$ up to $2 \times 10^2$ and $|e_1|$ up to $3 \times 10^3$ inside the return of $v$. That is harmless at $\eta = 0$, but it limits the height range over which $E = E_0 + \eta E_1 + \dots$ stays positive to roughly $|\eta| \lesssim |e_1|^{-1}$, and so becomes a quantitative input to (G) rather than a defect of the midplane.

**Exact statements this adds. (i)** In the class $E_1 = 0$ a stress-free global preparation is impossible: Theorem 9.22 with $\tilde{q} = 0$. **(ii)** With $E_1 \neq 0$, the obstruction of Lemma 9.12 is lifted exactly by $v\,e_1 = 2l' - h$ at the crossing, and the lift is linear everywhere else.

**Obstruction tests. (A)** *Compact support of $v$*: not ruled out. The bridge is exact up to the crossing, and a reversal after $l < 0$ costs no (CROSS); the test succeeds only against slaved returns (Theorem 9.22). **(B)** *Source rank*: the angular lift has no rank defect (Theorem 9.21). The axial demand has no independent control when $E_1 = 0$ — that is Theorem 9.22 — and regains one when $E_1 \neq 0$, (g). **(C)** *Moments*: no conflict. $\Psi(X_a) = 2 + \alpha$ is automatic, and at $\eta = 0$ both $c_Q$ and $c_N$ reduce to $w(X_a) = 0$. **(D)** *Join*: no conflict; $l = -h$ and $v = b_s = w = 0$ hold flatly at $X_a$ by construction. **(E)** *Cone rotation*: impossible on on-axis segments; realized numerically in every tested return window, with the mechanism of (f); not a theorem. **(F)** *Regularity*: $E = H/\sqrt{2X} > 0$ always; $v_{\mathrm{AX}}$ is the regular branch at a regular singular point with exponents $(0, 0)$; $b_s = O(X^{1/2})$ at the axis; $U^{(1)}(0) = 1 + h$. No axis singularity.

**Dependency graph.**

$$\underbrace{l(0^+) = 1,\ v(0) \neq 0}_{\text{axis}} \rightarrow \underbrace{w \equiv 0,\ E_1 \neq 0,\ v\,e_1 = 2l' - h \text{ at } l = 0}_{\text{stress-free preparation, Thms 9.25–9.26 under (NR)}} \rightarrow \underbrace{v = E_1 = 0,\ \Psi = 2 + \alpha}_{X_s \le X_a} \rightarrow \underbrace{\text{Thms 9.13–9.15}}_{\text{active layer}} \rightarrow \underbrace{v_s - 2 = \alpha}_{\text{exterior}}$$

The on-axis route of (c)–(g) remains a valid partial construction in the class $E_1 = 0$, and Theorem 9.22 remains its exact limit.

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## 10. Comparison with an existing construction [COND-OA]

OpenAI's *Finite Time Blowup for Navier–Stokes* [2] constructs a solution of this architecture. **The reduction of §§6–8 was derived after this construction was examined (§1.4, steps 5–6), so what follows is a consistency check and cannot be read as a prediction.** It has not yet been independently reviewed, and nothing below asserts that it is correct.

### 10.1 The mapping [MAP]

The construction has an axisymmetric background carrying a self-similar collapsing vortex, plus spatially oscillatory non-axisymmetric pulses of zero angular mean, ringing the axis in an annulus at the core's edge, whose averaged quadratic fluxes cancel the singular part of the background residual. Its stated scalings, with $\tau = 1 - t$ and $0 < h < 1/100$ fixed:

$$\ell_r \asymp \tau^{1/2}, \quad \ell_z \asymp \tau^{1/2\,-\,h}, \quad |u_\theta| \asymp |u_z| \asymp \tau^{-1/2\,-\,h}, \quad \mathrm{Re}_\theta \asymp \tau^{-h}, \quad \mathrm{Re}_r = O(1), \quad E_{\mathrm{core}} \asymp \tau^{1/2\,-\,3h}.$$

Identifying $\rho \leftrightarrow \ell_r$, $\ell \leftrightarrow \ell_z$, $\Lambda \leftrightarrow \ell_r|u_\theta|$ — these are **[MAP]**, not identities — the exterior profile is computed from the construction's own far-field form to be $\tau$-independent with $\Gamma_{\mathrm{ext}} \sim r^{-2h}$, so

$$\boxed{\alpha = 2h \;<\; \frac{1}{50}\,.}$$

### 10.2 The checks [EXACT | MAP]

| relation | § | value in the construction | result |
|---|---|---|---|
| $\mathcal{T}[\tau]$ as a radial flux divergence | 4 | its azimuthal relation is (4.1)'s radial term exactly | **independent confirmation** |
| $K = 2 + c/2\nu$, constant, $> 2$ | 2 | $\rho^2 \asymp c\tau$; contraction | ✓ |
| $G = \alpha(1 - 2/K) > 0$ | 7.1 | small and positive | ✓ |
| $\eta = -(1-\alpha)(1-2/K) < 0$ | 7.2 | $\ell \asymp \tau^{1/2\,-\,h}$ with $D = \frac{1}{2} - \frac{\alpha}{2}$ | ✓ |
| $a = \alpha/2$ | 7.3 | $a = h$ | ✓ **exactly** |
| (K3): $K(1 + \eta - G) = 2$ | 7.3 | — | ✓ **identically** |
| $E_{\mathrm{core}} \asymp \Lambda^2 \ell$ | 8.1 | $= \tau^{1/2\,-\,3h}$, matching the source's own figure | ✓ (no logarithm; the exterior circulation decays) |
| window $\alpha < \frac{1}{3}$ | 8.1 | $2h < 1/50$ | ✓ with a factor ~17 to spare |
| $\mathcal{T}[\tau] = 0$ on the core | 5.2 | the stress vanishes identically for $X \le X_a$ | ✓ — the load-bearing test's premise, realized |
| Lemma H at $\alpha = 2h$ | 9 | answered affirmatively by construction | **non-empty** |

> **What the agreement is, and is not.** It is **not** an independent convergence for the rows that matter most here: $G$, $\eta$, $a = \alpha/2$ and the window were derived with this construction in hand (§1.4). It **is** an independent convergence for the *architecture* and the $(K, G, \eta)$ ledger, which were reached on 10–11 September 2026 before the construction was read — but those enter this paper as background, not as its results. And for every row it is a real consistency check: the reduction is not vacuous, its exact relations hold for an object built on entirely different premises, and Lemma H has at least one solution. **No priority is claimed for any shared scaling, and none is available.**

### 10.3 One value the comparison adds

The construction's pulses live at a wavelength $\ell_{\text{wave}} \asymp \tau^{1/2\,+\,h/2}$, finer than the core radius by $\tau^{h/2}$, each growing and then decaying while new ones appear on successively finer scales. So the interaction field is not merely a perturbation of the core: it is a **finer subsystem**, of a different category from its parent, terminating individually while the sequence continues. Whether that is required by the architecture or particular to this construction is open, and it is the only place where the present reduction has nothing to say.

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## 11. Summary, and the dependency graph

$$\underbrace{(\mathrm{W})}_{\S2} \to \underbrace{(\mathrm{M}),\ \text{two torques}}_{\S3} \to \underbrace{(S')}_{\S4} \to \underbrace{(\mathrm{A1b})+\text{load-bearing test}}_{\S5} \to \underbrace{\Lambda \le \Gamma_{\text{ext}}(\rho)}_{\S6} \to \underbrace{(G,\eta,a)}_{\S7} \to \underbrace{0<\alpha<\tfrac{1}{3}}_{\S8}$$

and, branching from $\alpha$ and the profile rather than continuing the chain,

$$\alpha, \Phi \to \underbrace{(\alpha_{\text{loc}}, b_s)}_{\S9.4} \to s \to p_s \to w = p_s - s \to \underbrace{(M_0, M_1, M_2)}_{\S9.6} \to \underbrace{\mathcal{M}^{-1}(0) \cap \mathcal{U}}_{\S9.8\text{–}9.10}$$

with the moment closure entering as the **stress return to zero** (§9.6) rather than as a separate condition.

Every node is exact given the premises named at it. The premises are (A1)–(A5) and (A1b), the equations themselves, and — for §6 — the swirl maximum principle in the restricted form §3 establishes. **(A4) carries more than its length suggests**: it is what turns the one-sided bound (6.1) into the equality that §6.2(ii), §7 and §8 all differentiate. Remove (A4) and the reduction has an upper bound on Λ and nothing else. Remove (A5) and §7.2, §7.3 and §8 go; remove (A1b) and §6 onward goes, which is the substance of §5.3.

**What would falsify the reduction.** An instance of this architecture with $\alpha \ge \frac{1}{3}$ and finite energy; or with $\eta > 0$ under (A5); or with $\mathcal{T}[\tau]$ load-bearing inside the core — a violation of (A1b), which would restore the eigenvalue problem §5.3 describes and with it the exclusion. What that would falsify is the class premise (A1b), not the classification rule R9. For §9 specifically: a profile whose source terms carry a contribution the transcription of §9.9 missed, which would change the moment weights and require Theorem 9.6 to be recomputed. Any of these would locate the failure at a named premise rather than in the chain.

**What is open.** The global midplane problem first, then the matching in height. Lemma $H''$ is settled **on the active layer** of the fibre $\eta = 0$, analytically and **[EXACT | TRANSCRIPTION]** (Theorems 9.13–9.15). But Theorem 9.18 shows that with zero midplane axial velocity the preparation region inboard of the layer necessarily carries unrealizable demand. The first open problem is therefore a midplane profile with $U(\cdot, 0) \not\equiv 0$ inboard whose demand is zero or cone-admissible at every radius; the reflection-symmetric class is excluded. §9.16.9 closes it. With $\partial_\eta E|_0 = 0$ the preparation lifts exactly and is on-axis admissible through the crossing (Theorems 9.21, 9.24), but it cannot be on-axis all the way to the layer (Theorem 9.22). With $\partial_\eta E|_0 \neq 0$ the whole inboard region is made **stress-free** by a linear lift (Theorem 9.25). Theorem 9.26 proves that lift uniquely solvable whenever the midplane axial velocity exceeds an explicit threshold on the inner region, $\min v^2 > \int_0^{X_0} E_0^2\, l/x$; below that threshold resonances occur, at a discrete sequence of amplitudes. **The global midplane is closed [EXACT | TRANSCRIPTION].** What remains open for Lemma $H''$ is the matching in height, and §9.16.8(h) settles its framework. The general endpoint/moment system (G) has rank four; within the stress-free inboard class it is two scalar outer-edge conditions plus one functional stress-free constraint. That constraint is parabolic in $\eta$, with direction set by the axial velocity (Theorem 9.27), so the closed midplane is not initial data. Nor is the stagnation surface $H_c = 0$ a design manifold: it carries no intrinsic trace where it is not horizontal, a horizontal stagnation slice is excluded

inboard (Theorem 9.29), and both ends of the strip are outflow (Proposition 9.28). The formal $\eta$-hierarchy closes to every order with no new resonance, but it is of heat type and is not a construction. Nor does an intrinsic clock remove the difficulty. With $\tau_\eta = 1/H_c$ each sign-domain is forward-parabolic along its own drift, and the stagnation set lies at infinite intrinsic depth along drift lines (Proposition 9.30). But the diffusion leaves $\eta =$ const cross any non-horizontal $\Sigma_*$ at finite radial distance, so the two sign-domains separate only where $\Sigma_*$ is horizontal (Theorem 9.31), and inboard they form one system. Nor can its singular set be designed down to one point. Sinks and leaf tangencies are one condition on the direction of $\nabla H_c$ (Proposition 9.32). Neither is forced topologically, but a fold tangency at the axis and a flat junction at $(X_s, 0)$ are (Theorem 9.33), and near-midplane sinks appear in every tested closed-midplane lift **[NUM]**. The geometry of $\Sigma_*$ is state of the solution, steerable only through the handover trace. The scalar energy condition ($\mathrm{NR}_\eta$) turned out to be the wrong quantity. In the full linearised system the radial drift cancels $U_\eta$ from the transport divergence, and no bounded positive local diagonal multiplier symmetrizes the $\eta$-derivative couplings (Theorem 9.34). The height mass operator is the Theorem 9.26 operator on every leaf, with infinite negative index on the leaf through the forced axis fold (Theorem 9.35), and that index is not an instability. The scalar axis fold is well posed and hypoelliptic (Theorem 9.36). At the fold point the swirl–pressure pair sets a quasi-elliptic $(2, 6)$ core, with an inherited complementing radial triple (Theorem 9.37). Core and outer fold are the same two modes, with eigenvectors independent of the transition parameter (Theorem 9.38). The mode zeros reduce to one scale-invariant normal form $\mathcal{N}_{\mathrm{Ri}}$, whose exponent is the leafwise Frobenius exponent (Theorem 9.39). That normal form is exactly the scalar Kolmogorov equation in the nonlocal gauge $|D|^c$, with an energy identity of constant 1 and no data at the zero (Theorem 9.40). At the axis the symmetrization excluded in general by Theorem 9.34 is admissible, with weights degenerating like $E^2$. $H_c$ plus the swirl–pressure pair is then self-adjoint, and one energy covers core, corner and outer fold: the symmetric fold is well posed on small windows, with no height data and no resonance (Theorem 9.41). The shear's skew Volterra mass cannot be absorbed by any approximate symmetrizer (Lemma 9.43), so the glued route is [FAIL / REDUNDANT]. At operator level the fold is uniformly invertible at sufficiently small height frequency (Proposition 9.44). At large height frequency the local models have trivial kernels on their specified domains (Proposition 9.46); global exhaustion requires a separate proof. At the axis the modes decouple exactly (Theorem 9.47), the fold mass has no kernel (Lemma 9.48), and quasimodes of bounded energy are impossible (Proposition 9.49(a)). Lemma 9.50 now proves a uniform lower bound for the whole-line normal form even at zero Fourier frequency. The unrestricted weighted Green operator fails through range compatibility, not through quasimodes. Proposition 9.51 proves that every hypothetical quasimode concentrates at the axis. The global argument identifies a proposed transport limit with a pressure-induced reflection relation (Proposition 9.52); that reflected model is uniformly invertible (Lemma 9.53), whereas dropping the relation admits a kernel. Proposition 9.54 transfers the reflection to actual viscous primitive coefficients on a shrinking outer collar, with vanishing normalized error. Proposition 9.55 identifies the local transport limits. Proposition 9.56 proves they inherit the pressure reflection and therefore vanish; Proposition 9.57 removes the physical norm outside every fixed fraction of the transport scale. **Theorem 9.60 closes the full frozen-window high-frequency compactness statement, including (C*) and the stated large-frequency target (S'a).** Proposition 9.58 controls every fixed $|\tau|^{-1/6}$ core; Lemma 9.59 controls the exact six-component state between that core and the transport scale, with both endpoint states justified. The original physical norm is exhausted. No numerical growth rate or all-frequency inverse is asserted. Surjectivity, the source-fold term $\kappa t$, height outflow and global weak inboard patching remain subsequent obligations. No failure of the PDE has been proved, and no obstruction test reached it. Off the midplane, §9.16.8 settles the corner question locally. Moving edges are not forced: Theorem 9.17 gives fixed edges on full rectangles for circulation-driven edge zones. They are what every finite truncation of the axial-flux-driven continuation produces, for a structural reason — three radial derivatives per order in height — that leaves truncations unable to decide the question. Theorem 9.17 is

conditional on the four matching identities $w(X_a, \eta) = w(X_b, \eta) = 0$ and $c_Q(\eta) = c_N(\eta) = 0$; they are solved at first order, where Corollary 9.16 forces the pressure to bend axially, and remain open beyond it. Beyond that, the two scope questions of §6.2(iii) and §10.3 remain. Surjectivity of the moment map is settled (§9.9), and the moments are exactly solvable (§9.11). The cone half of realizability is solved (§9.3), its parameter is identified with the window exponent (§9.4), uniformity is structural (§9.5), the problem is shown to be a design problem rather than a trajectory problem (§9.6), and the moment repair is quantified (§9.8).

### 11.1 What the method contributed, and what it cost

**Contributed.** Two steps. R9 classifying $\mathcal{T}[\tau]$ as a matching-problem term, given the source-free-core premise (A1b) (§5.2); the supply theorem, the kinematic sign of $\eta$, the cancellation of $K$ and the replacement of an eigenvalue problem by two formulas all follow from it. And R10 keeping the stress a **pair**, without which §9 would have asked a scalar-sign question of a two-dimensional cone and reached no theorem. Two smaller contributions: R11 identifying $K$ as cut-defined (§8.2), and R15 reducing the decomposition from six subsystems to three (§5.1).

**A second flattening, caught by a failed proof.** §9.6 records it: having removed the false core-stress problem, the analysis nearly replaced the surviving profile problem with a finite radial state and attempted an invariant-region proof on it. The attempt could not close, and the reason is the theorem — **radial state $\neq$ profile state**. The rules did not prevent this in advance; the proof attempt exposed it. They diagnose reliably and predict only sometimes.

**Cost.** Before the rules were written down, the same programme produced the core-driven reduction and an exclusion that did not hold (§5.3); the rules were formalized in response to that failure and others like it (§1.4). The programme also suggested the annulus localization of §6.2(iii), which could not be proved and is marked accordingly. And until §5.2 was repaired, a classification rule stood in for a dynamical premise: the vanishing of the stress on the core was presented as an output of R9 when it is the class premise (A1b). Every result above is therefore stated with its premises attached rather than with the authority of the method.

**The falsifiable form of the claim.** If the decomposition of §5 is wrong for a given candidate, some term placed here in the matching relation is load-bearing in that candidate's core. That is a violation of (A1b), checkable directly in (S′), and it would restore the eigenvalue problem and the exclusion for that candidate.

**The form the claim should take.**

| The protocol identified a representational error that had produced correct mathematics about the wrong |
|---|

The core-driven reduction of §5.3 is not erroneous arithmetic. Its eigenvalue is the correct eigenvalue of the problem it poses, a core sustained by an internal stress. What was wrong was the identification of that problem with this architecture: a misassignment of one coupling term, invisible to any check internal to the calculation, and visible to a rule about where couplings live.

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## Notes on sources

The width law, the identity (M) and (S′) are elementary and are verified here by direct computation. The swirl maximum principle for axisymmetric flow is classical; see [4], and [5] for the parabolic maximum principle. The construction discussed in §10 is OpenAI's *Finite Time Blowup for Navier–Stokes* [2], with its accompanying Lean formalization [9]. It is the only source read in full against the primary document, and it has not yet been independently reviewed. Its §4 supplies the covariance-cone inequalities and the profile definitions that §9 computes with. The transcription of the source terms $S_q, S_n$ is the one place where a

misreading would change a stated result, and §9.9 says so at its own site. D. Gondauri's critical audit of that construction [10] is used twice in §9: its flag that squaring the cone inequality could lose sign information is checked and discharged, and its objection that a pointwise gap need not be uniform on a noncompact domain is what §9.5 answers. That audit reports no defect and does not independently discharge the construction's proof obligations; nothing here treats it as doing so.

The decomposition rules of §1.3 are stated in [12]. They supply no mathematics, and no conclusion here depends on them.

The reformulation of blow-up as an event count and the exact circulation cost per turnover, which motivated this line of work, are not used and are reported separately. The earlier core-driven reduction of §5.3, with its numerical eigenvalue and the exclusion it produced, belongs to the author's earlier unpublished work; no result here depends on it.